

Spectral analysis of matrices in isogeometric collocation methods

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Abstract

We consider a linear full elliptic second order partial differential equation in a d -dimensional domain, $d \geq 1$, approximated by isogeometric collocation methods based on uniform B-splines of degrees $\mathbf{p} := (p_1, \dots, p_d)$, $p_j \geq 2$, $j = 1, \dots, d$. We give a construction of the inherently non-symmetric matrices arising from this approximation technique and we perform an analysis of their spectral properties. In particular, we find the associated (spectral) symbol, that is the function describing their asymptotic spectral distribution (in the Weyl sense), when the matrix-size tends to infinity or, equivalently, the fineness parameters tend to zero. The symbol is a nonnegative function with a unique zero of order two at $\boldsymbol{\theta} = \mathbf{0}$ (with $\boldsymbol{\theta}$ the Fourier variables), but with infinitely many numerical zeros for large $\|\mathbf{p}\|_\infty$, showing up at $\theta_j = \pi$ if p_j is large. The presence of a zero of order two at $\boldsymbol{\theta} = \mathbf{0}$ is expected, because it is intrinsic in any local approximation method of differential operators, like finite differences and finite elements. However, the second type of zeros leads to the surprising fact that, for large $\|\mathbf{p}\|_\infty$, there is a subspace of high frequencies where the collocation matrices are ill-conditioned. This non-canonical feature is responsible for the slowdown, with respect to \mathbf{p} , of standard iterative methods. On the other hand, its knowledge and the knowledge of other properties of the symbol can be exploited to construct iterative solvers with convergence properties, independent of the fineness parameters and of the degrees \mathbf{p} .

Keywords : Spectral distribution, symbol, collocation method, B-splines, isogeometric analysis.

SPECTRAL ANALYSIS OF MATRICES IN ISOGEOMETRIC COLLOCATION METHODS

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ABSTRACT. We consider a linear full elliptic second order partial differential equation in a d -dimensional domain, $d \geq 1$, approximated by isogeometric collocation methods based on uniform B-splines of degrees $\mathbf{p} := (p_1, \dots, p_d)$, $p_j \geq 2$, $j = 1, \dots, d$. We give a construction of the inherently non-symmetric matrices arising from this approximation technique and we perform an analysis of their spectral properties. In particular, we find the associated (spectral) symbol, that is the function describing their asymptotic spectral distribution (in the Weyl sense), when the matrix-size tends to infinity or, equivalently, the fineness parameters tend to zero. The symbol is a nonnegative function with a unique zero of order two at $\boldsymbol{\theta} = \mathbf{0}$ (with $\boldsymbol{\theta}$ the Fourier variables), but with infinitely many numerical zeros for large $\|\mathbf{p}\|_\infty$, showing up at $\theta_j = \pi$ if p_j is large. The presence of a zero of order two at $\boldsymbol{\theta} = \mathbf{0}$ is expected, because it is intrinsic in any local approximation method of differential operators, like finite differences and finite elements. However, the second type of zeros leads to the surprising fact that, for large $\|\mathbf{p}\|_\infty$, there is a subspace of high frequencies where the collocation matrices are ill-conditioned. This non-canonical feature is responsible for the slowdown, with respect to \mathbf{p} , of standard iterative methods. On the other hand, its knowledge and the knowledge of other properties of the symbol can be exploited to construct iterative solvers with convergence properties, independent of the fineness parameters and of the degrees \mathbf{p} .

1. INTRODUCTION

Isogeometric Analysis (IgA) is a paradigm for the analysis of problems governed by partial differential equations [10]. Its goal is to improve the connection between numerical simulation and Computer Aided Design (CAD) systems. In its original formulation, the main idea in IgA is to use directly the geometry provided by CAD systems and to approximate the unknown solutions of differential equations by the same type of functions. Tensor-product B-splines and their rational extension, the so-called NURBS, are the dominant technology in CAD systems used in engineering, and thus also in IgA. Here we consider the following linear full elliptic second order partial differential equation (PDE), with non-constant coefficients and homogeneous

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Key words and phrases. Spectral distribution, symbol, collocation method, B-splines, isogeometric analysis.

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Dirichlet boundary conditions:

$$(1.1) \quad \begin{cases} -\nabla \cdot K \nabla u + \boldsymbol{\alpha} \cdot \nabla u + \gamma u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in \mathbb{R}^d , $K : \overline{\Omega} \rightarrow \mathbb{R}^{d \times d}$ is a symmetric positive definite (SPD) matrix of C^1 functions, $\boldsymbol{\alpha} : \overline{\Omega} \rightarrow \mathbb{R}^d$ is a vector of continuous functions, $\gamma \geq 0 \in C(\overline{\Omega})$ and $f \in C(\overline{\Omega})$.

Galerkin formulations have been intensively employed in the IgA context. In such a case, an efficient implementation requires special numerical quadrature rules when assembling the system of equations, see e.g. [16]. To avoid this issue, isogeometric collocation methods have been recently introduced and used in [2, 21]. We briefly describe the isogeometric collocation approach in Section 2.1, where uniform B-splines of arbitrary degrees $\mathbf{p} := (p_1, \dots, p_d)$, $p_j \geq 2$, $j = 1, \dots, d$, are considered.

In this paper we focus on the study of the spectrum of the resulting collocation IgA matrices approximating problem (1.1), when the fineness parameters tend to zero and the related matrix-size tends to infinity. More specifically, besides the conditioning and the extremes of the spectrum (see e.g. [17, 26, 27]), which are important for evaluating the inherent error, we are interested whether there exists an eigenvalue distribution in the Weyl sense [8]. As already demonstrated in several contexts, such an information plays a role in the convergence analysis of (preconditioned) Krylov methods (see [3, 18] and references therein) and is even more important in the design and in the theoretical analysis of effective preconditioners [22, 26] and effective multigrid/multi-iterative solvers [1, 11]. Here we prove the following:

- a) an eigenvalue distribution exists and is compactly described by a symbol f ;
- b) the symbol f has a canonical structure incorporating
 - b1) the approximation technique, which is identified by a finite set of polynomials in the Fourier variables $\boldsymbol{\theta} := (\theta_1, \dots, \theta_d) \in [0, \pi]^d$;
 - b2) the geometry, which is identified by the map \mathbf{G} in the variables $\hat{\mathbf{x}} := (\hat{x}_1, \dots, \hat{x}_d)$ defined on the reference domain $[0, 1]^d$;
 - b3) the coefficients of the principal terms of the PDE, namely K , in the physical variables $\mathbf{x} := (x_1, \dots, x_d)$ defined on the physical domain Ω .

In reality, the picture in item b) is intrinsic to the approximation of PDEs by any local method, such as Finite Differences (FDs) and Finite Elements (FEs). In fact, formally, the structure of the symbol is substantially the same when considering different techniques (see [4, 24] and references therein) to approximate the same problem. The only difference is due to the polynomials in the Fourier variables and this is no surprise, since this part specifically depends on the chosen approximation technique. Even though the formal structure of the symbol is not new, some of the analytic features of the symbol are not expected. For instance, when one of the \mathbf{p} parameters, say p_j , becomes large, then the symbol f shows a numerical zero at $\theta_j = \pi$ and the convergence to zero is monotonic and exponential with respect to p_j . The latter information implies that small eigenvalues appear related to high frequency eigenvectors and this non-canonical source of ill-conditioning is indeed responsible for a sensible slowdown of all the standard multigrid and preconditioning techniques. On the other hand, the latter information can be exploited for designing ad hoc algorithms with convergence speed independent of the fineness parameters and the approximation parameters \mathbf{p} . We refer to [11] for extensive numerical

results and [13] for the same approach in the Galerkin B-spline IgA setting, both in the simplified case where K is the identity matrix.

Finally, it is worthwhile to emphasize the mathematical tools used in our derivation. We use the theory of Generalized Locally Toeplitz (GLT) sequences, which goes back to the pioneering work by Tilli [28] and is developed in [24, 25], and, implicitly, the concept of approximation class of sequences (a.c.s.), see [23], which allows us to derive the spectral distribution of a complicated sequence of matrices (matrix-sequence) starting from those of simpler matrix-sequences. We also exploit general results, contained in [15] and generalized in [14], which allow us to determine the spectral distribution of arbitrary (non-Hermitian) perturbations of Hermitian matrix sequences, under certain conditions on the perturbation matrices. Here we do not use the quoted machinery in its full generality, but, as done in [4, 19], we somehow simplify the approach, since the considered matrices show a band structure in a multilevel sense.

The paper is organized as follows. In Section 2 we introduce our model problem, we define the collocation method, and we present preliminary tools for the spectral analysis. Section 3 deals with the symbols associated to cardinal B-splines, which are used in Sections 4 and 5 to give a formal expression of the associated collocation matrices and of their spectral distributions in 1D and 2D, respectively. We also provide the expression of the general symbol in the d -dimensional setting. We end in Section 6 with conclusions and open problems. An extensive numerical testing of fast solvers, designed in accordance with the given spectral analysis, is reported and discussed in the twin paper [12].

2. NOTATIONS AND TOOLS

Problem (1.1) can be reformulated as follows:

$$(2.1) \quad \begin{cases} -\mathbf{1}(K \circ Pu)\mathbf{1}^T + \beta \cdot \nabla u + \gamma u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\mathbf{1} := [1 \cdots 1]$, Pu denotes the Hessian of u , i.e.

$$(Pu)_{i,j} := \frac{\partial^2 u}{\partial x_i \partial x_j},$$

and \circ denotes the componentwise Hadamard matrix product (see [5]). Moreover, β collects the coefficients of the first order derivatives in (1.1), namely

$$\beta_j := \alpha_j - \sum_{i=1}^d \frac{\partial \kappa_{i,j}}{\partial x_i},$$

with $\kappa_{i,j}$ the entries of the matrix K .

We consider an approximation of the solution of the problem (2.1) by the standard collocation approach, as explained briefly in the following. Let \mathcal{W} be a finite dimensional approximation space of sufficiently smooth functions vanishing on the boundary. Then, we introduce a set of $N := \dim \mathcal{W}$ collocation points in Ω ,

$$(2.2) \quad \{\tau_i \in \Omega, \quad i = 1, \dots, N\},$$

and we look for a function $u_{\mathcal{W}} \in \mathcal{W}$ such that

$$(2.3) \quad -\mathbf{1}(K(\tau_i) \circ Pu_{\mathcal{W}}(\tau_i))\mathbf{1}^T + \beta(\tau_i) \cdot \nabla u_{\mathcal{W}}(\tau_i) + \gamma(\tau_i)u_{\mathcal{W}}(\tau_i) = f(\tau_i), \quad \forall \tau_i.$$

If we fix a basis $\{\varphi_1, \dots, \varphi_N\}$ for \mathcal{W} , then each $v \in \mathcal{W}$ can be written as $v = \sum_{j=1}^N v_j \varphi_j$. The collocation problem (2.3) is equivalent to the problem of finding a vector $\mathbf{u} := [u_1 \ u_2 \ \dots \ u_N]^T \in \mathbb{R}^N$ such that

$$(2.4) \quad A\mathbf{u} = \mathbf{f},$$

where

$$A := [-\mathbf{1}(K(\boldsymbol{\tau}_i) \circ P\varphi_j(\boldsymbol{\tau}_i))\mathbf{1}^T + \boldsymbol{\beta}(\boldsymbol{\tau}_i) \cdot \nabla \varphi_j(\boldsymbol{\tau}_i) + \gamma(\boldsymbol{\tau}_i)\varphi_j(\boldsymbol{\tau}_i)]_{i,j=1}^N \in \mathbb{R}^{N \times N}$$

is the collocation matrix and $\mathbf{f} := [f(\boldsymbol{\tau}_i)]_{i=1}^N$. Once we find \mathbf{u} , we know $u_{\mathcal{W}} = \sum_{j=1}^N u_j \varphi_j$. The regularity of the system (2.4) depends on the selection of the collocation points (2.2).

2.1. Isogeometric collocation methods. Let

$$(2.5) \quad \{\hat{\varphi}_1, \dots, \hat{\varphi}_{N+N_b}\}$$

be a set of basis functions defined on $\hat{\Omega} := [0, 1]^d$, such that the physical domain Ω in (2.1) can be described by a global geometry function,

$$(2.6) \quad \mathbf{G} : \hat{\Omega} \rightarrow \overline{\Omega}, \quad \mathbf{G}(\hat{\mathbf{x}}) := \sum_{i=1}^{N+N_b} \hat{\varphi}_i(\hat{\mathbf{x}}) \mathbf{p}_i, \quad \mathbf{p}_i \in \mathbb{R}^d, \quad \hat{\mathbf{x}} \in \hat{\Omega}.$$

We assume that the map \mathbf{G} is invertible in $\hat{\Omega}$ and $\mathbf{G}(\partial\hat{\Omega}) = \partial\overline{\Omega}$. If $\{\hat{\varphi}_1, \dots, \hat{\varphi}_N\}$ is defined as the subset of functions given in (2.5) which vanish on the boundary $\partial\hat{\Omega}$, then the approximation space \mathcal{W} is spanned by

$$(2.7) \quad \varphi_i(\mathbf{x}) := \hat{\varphi}_i(\mathbf{G}^{-1}(\mathbf{x})) = \hat{\varphi}_i(\hat{\mathbf{x}}), \quad i = 1, \dots, N, \quad \mathbf{x} = \mathbf{G}(\hat{\mathbf{x}}).$$

In the most common formulation of IgA, the functions in (2.5) are tensor-product B-splines or NURBS, since they allow an exact representation – by definition – of an arbitrary domain designed in a (NURBS-based) CAD system. Nonetheless, other kinds of functions can be used as well.

Finally, we introduce a set of collocation points in the parametric domain,

$$\{\hat{\boldsymbol{\tau}}_j \in \hat{\Omega}, \quad j = 1, \dots, N\},$$

and we set

$$(2.8) \quad \boldsymbol{\tau}_j := \mathbf{G}(\hat{\boldsymbol{\tau}}_j).$$

In the isogeometric collocation approach, we solve the linear system (2.4) with the basis functions and the collocation points given by (2.7) and (2.8), respectively.

In this paper we present a detailed spectral analysis of the matrices obtained by isogeometric collocation methods based on B-splines with equally spaced knots for problem (2.1) in the univariate and in the bivariate setting. We first address the case of the unit interval and the unit square, and then we investigate the case of more complex geometries.

The choice of the collocation points is crucial for the stability and good behavior of the discrete problem. Following the approach in [2], our collocation points in the parametric domain are taken as the Greville abscissae corresponding to the used B-splines.

2.2. Preliminaries on spectral analysis. Let us start with introducing some linear algebra notation and recalling some basic results that will be used throughout this paper. We refer to [5] for more details on basic linear algebra results.

Given a matrix $X \in \mathbb{C}^{m \times m}$, $\|X\|$ is the 2-norm of X , i.e. $\|X\| = \sqrt{\rho(X^*X)} = s_1(X)$, where $s_1(X)$ is the maximum singular value of X and $\rho(X)$ is the spectral radius of X . Denote by $\|X\|_1$ the trace norm of X , i.e. the sum of all the singular values of X : $\|X\|_1 = \sum_{j=1}^m s_j(X)$. Since the number of nonzero singular values of X is precisely $\text{rank}(X)$, it follows that $\|X\|_1 \leq \text{rank}(X)\|X\| \leq m\|X\|$.

We now introduce the fundamental definitions for developing our spectral analysis, see [15, Definitions 1.1 and 1.2]. We denote by μ_d the Lebesgue measure in \mathbb{R}^d .

Definition 2.1 (Spectral distribution of a sequence of matrices). Let $C_c(\mathbb{C}, \mathbb{C})$ be the space of continuous functions $F : \mathbb{C} \rightarrow \mathbb{C}$ with compact support. Let $\{X_n\}$ be a sequence of matrices with increasing dimension ($X_n \in \mathbb{C}^{d_n \times d_n}$ with $d_n < d_{n+1}$ for every n), and let $f : D \rightarrow \mathbb{C}$ be a measurable function defined on the measurable set $D \subset \mathbb{R}^d$ with $0 < \mu_d(D) < \infty$. We say that $\{X_n\}$ is distributed like f in the sense of the singular values and we write $\{X_n\} \stackrel{s}{\sim} f$, if

$$\lim_{n \rightarrow \infty} \frac{1}{d_n} \sum_{j=1}^{d_n} F(\sigma_j(X_n)) = \frac{1}{\mu_d(D)} \int_D F(|f(x_1, \dots, x_d)|) dx_1 \cdots dx_d, \quad \forall F \in C_c(\mathbb{C}, \mathbb{C}).$$

Similarly, we say that $\{X_n\}$ is distributed like f in the sense of the eigenvalues and we write $\{X_n\} \stackrel{\lambda}{\sim} f$, if

$$\lim_{n \rightarrow \infty} \frac{1}{d_n} \sum_{j=1}^{d_n} F(\lambda_j(X_n)) = \frac{1}{\mu_d(D)} \int_D F(f(x_1, \dots, x_d)) dx_1 \cdots dx_d, \quad \forall F \in C_c(\mathbb{C}, \mathbb{C}).$$

Definition 2.2 (Clustering of a sequence of matrices at a subset of \mathbb{C}). Let $\{X_n\}$ be a sequence of matrices with increasing dimension ($X_n \in \mathbb{C}^{d_n \times d_n}$ with $d_n < d_{n+1}$ for every n), and let $S \subseteq \mathbb{C}$ be a non-empty closed subset of \mathbb{C} . We say that $\{X_n\}$ is *strongly clustered* at S if the following condition is satisfied:

$$\forall \varepsilon > 0, \quad \exists C_\varepsilon \text{ and } \exists n_\varepsilon : \quad \forall n \geq n_\varepsilon, \quad q_n(\varepsilon) \leq C_\varepsilon,$$

where $q_n(\varepsilon)$ is the number of eigenvalues of X_n lying outside the ε -expansion S_ε of S , i.e.,

$$S_\varepsilon := \bigcup_{s \in S} [\text{Re } s - \varepsilon, \text{Re } s + \varepsilon] \times [\text{Im } s - \varepsilon, \text{Im } s + \varepsilon].$$

We say that $\{X_n\}$ is *weakly clustered* at S if the quantity $q_n(\varepsilon)$ is $o(d_n)$ as $n \rightarrow \infty$, i.e.,

$$\lim_{n \rightarrow \infty} \frac{q_n(\varepsilon)}{d_n} = 0.$$

We also recall the following results, see [15, Theorems 3.4 and 3.5].

Theorem 2.3. Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of matrices with $X_n, Y_n \in \mathbb{C}^{d_n \times d_n}$, and $d_n < d_{n+1}$ for all n , such that

- X_n is Hermitian for all n and $\{X_n\} \stackrel{\lambda}{\sim} f$, where $f : D \subset \mathbb{R}^d \rightarrow \mathbb{R}$ is a measurable function defined on the measurable set D with $0 < \mu_d(D) < \infty$;
- there exists a constant C so that $\|X_n\|, \|Y_n\| \leq C$ for all n ;
- $\|Y_n\|_1 = o(d_n)$ as $n \rightarrow \infty$, i.e., $\lim_{n \rightarrow \infty} \frac{\|Y_n\|_1}{d_n} = 0$.

Set $Z_n := X_n + Y_n$. Then $\{Z_n\} \overset{\lambda}{\sim} f$ and $\{Z_n\}$ is weakly clustered at the essential range of f .¹

Theorem 2.4. Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of matrices with $X_n, Y_n \in \mathbb{C}^{d_n \times d_n}$, and $d_n < d_{n+1}$ for all n , such that

- X_n is Hermitian for all n and $\{X_n\} \overset{\lambda}{\sim} f$, where $f : D \subset \mathbb{R}^d \rightarrow \mathbb{R}$ is a measurable function defined on the measurable set D with $0 < \mu_d(D) < \infty$;
- there exists a constant C so that $\|X_n\|, \|Y_n\|_1 \leq C$ for all n .

Set $Z_n := X_n + Y_n$. Then $\{Z_n\} \overset{\lambda}{\sim} f$ and $\{Z_n\}$ is strongly clustered at the essential range of f .

Given the multi-index $\mathbf{m} := (m_1, \dots, m_d)$, the d -level Toeplitz matrix $T_{\mathbf{m}}(f)$ associated to a function $f \in L_1([-\pi, \pi]^d)$ is defined as

$$(2.9) \quad T_{\mathbf{m}}(f) := \left[\cdots \left[[f_{i_1-j_1, i_2-j_2, \dots, i_d-j_d}]_{i_d, j_d=1}^{m_d} \right]_{i_{d-1}, j_{d-1}=1}^{m_{d-1}} \cdots \right]_{i_1, j_1=1}^{m_1},$$

where for each $\mathbf{k} := (k_1, \dots, k_d) \in \mathbb{Z}^d$,

$$f_{\mathbf{k}} := \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(\theta_1, \dots, \theta_d) e^{-i(k_1\theta_1 + \dots + k_d\theta_d)} d\theta_1 \cdots d\theta_d$$

is the \mathbf{k} -th Fourier coefficient of f . The function f is called the *generating function* of the Toeplitz family $\{T_{\mathbf{m}}(f)\}_{\mathbf{m} \in \mathbb{N}^d}$. From the theory of Toeplitz matrices, we know that

$$(2.10) \quad \|T_{\mathbf{m}}(f)\| \leq M_f, \quad M_f := \operatorname{ess\,sup}_{\boldsymbol{\theta} \in [-\pi, \pi]^d} |f(\boldsymbol{\theta})|.$$

where $\operatorname{ess\,sup}_{\boldsymbol{\theta} \in [-\pi, \pi]^d} |f(\boldsymbol{\theta})| = \max_{\boldsymbol{\theta} \in [-\pi, \pi]^d} |f(\boldsymbol{\theta})|$, because in this paper we will only deal with continuous functions f . The above inequality is strict whenever $|f(\boldsymbol{\theta})|$ is not constant (see e.g. [8]). The d -level diagonal sampling matrix $D_{\mathbf{m}}(a)$ associated to a Riemann integrable function $a \in [0, 1]^d \rightarrow \mathbb{C}$ is defined as

$$(2.11) \quad D_{\mathbf{m}}(a) := \operatorname{diag}_{j_1=0, \dots, m_1-1} \left(\cdots \operatorname{diag}_{j_d=0, \dots, m_d-1} \left(a \left(\frac{j_d}{m_d}, \dots, \frac{j_1}{m_1} \right) \right) \cdots \right).$$

Furthermore, throughout the paper, when we use a sequence based on a multi-index $\mathbf{m} \rightarrow \infty$, we actually mean that $\min_{j=1, \dots, d} m_j \rightarrow \infty$.

The following theorem describes the asymptotic distribution of sequences of matrices obtained from algebraic operations on Toeplitz matrices and diagonal sampling matrices.

Theorem 2.5. Given $k \in \{1, \dots, \eta\}$ and $l \in \{1, \dots, \eta_k\}$, let $\{B_{\mathbf{m}}^{(k,l)}\}$ be a sequence of matrices, where

$$\{B_{\mathbf{m}}^{(k,l)}\} = \{D_{\mathbf{m}}(a^{(k,l)})\}, \quad \text{or} \quad \{B_{\mathbf{m}}^{(k,l)}\} = \{T_{\mathbf{m}}(f^{(k,l)})\},$$

and let

$$X_{\mathbf{m}} = \sum_{k=1}^{\eta} \prod_{l=1}^{\eta_k} B_{\mathbf{m}}^{(k,l)}.$$

¹Note that if $f : D \subset \mathbb{R}^d \rightarrow \mathbb{C}$ is continuous, then the essential range of f coincides with the closure of the range of f , provided that the domain D is ‘sufficiently regular’. Here, ‘sufficiently regular’ means that D is contained in the set of accumulation points of its interior \bar{D} .

Then,

$$\{X_{\mathbf{m}}\} \stackrel{\sigma}{\sim} \sum_{k=1}^{\eta} \prod_{l=1}^{\eta_k} b^{(k,l)},$$

where

$$b^{(k,l)}(x_1, \dots, x_d; \theta_1, \dots, \theta_d) = \begin{cases} a^{(k,l)}(x_1, \dots, x_d), & \text{if } \{B_{\mathbf{m}}^{(k,l)}\} = \{D_{\mathbf{m}}(a^{(k,l)})\}, \\ f^{(k,l)}(\theta_1, \dots, \theta_d), & \text{if } \{B_{\mathbf{m}}^{(k,l)}\} = \{T_{\mathbf{m}}(f^{(k,l)})\}. \end{cases}$$

Moreover, if $\{X_{\mathbf{m}}\}$ is a sequence of Hermitian matrices, then

$$\{X_{\mathbf{m}}\} \stackrel{\lambda}{\sim} \sum_{k=1}^{\eta} \prod_{l=1}^{\eta_k} b^{(k,l)}.$$

The above theorem combines results from [25, Theorem 2.2] and [24, Theorems 4.5 and 4.8]. These results are formulated in the more general setting of GLT sequences and are based on the a.c.s. notion given in [23], but already present in the seminal work by Tilli [28]. Theorem 2.5 could also be extended by including the (pseudo-)inverse of matrices under mild assumptions on the function $b^{(k,l)}$, namely that the set where $b^{(k,l)}$ vanishes has zero Lebesgue measure, see [25, Theorem 2.2].

We now focus on a specific application of Theorem 2.5 which will be of interest later on. Given a d -level diagonal sampling matrix $D_{\mathbf{m}}(a)$, we define the symmetric matrix $\tilde{D}_{\mathbf{m}}(a)$ as

$$(2.12) \quad (\tilde{D}_{\mathbf{m}}(a))_{i,j} := (D_{\mathbf{m}}(a))_{\min(i,j), \min(i,j)} = \begin{cases} (D_{\mathbf{m}}(a))_{i,i}, & \text{if } i \leq j, \\ (D_{\mathbf{m}}(a))_{j,j}, & \text{if } i > j. \end{cases}$$

In the next corollary, we relate a sequence of matrices of the form $\{\tilde{D}_{\mathbf{m}}(a) \circ T_{\mathbf{m}}(f)\}$ to the tensor-product function $a \otimes f$ defined over $[0, 1]^d \times [-\pi, \pi]^d$ by

$$(a \otimes f)(x_1, \dots, x_d; \theta_1, \dots, \theta_d) := a(x_1, \dots, x_d) f(\theta_1, \dots, \theta_d).$$

Corollary 2.6. *Let $\{T_{\mathbf{m}}(f)\}$ be a sequence of d -level Toeplitz matrices associated with a d -variate trigonometric polynomial f , and let $\{\tilde{D}_{\mathbf{m}}(a)\}$ be a sequence of matrices with $\tilde{D}_{\mathbf{m}}(a)$ defined in (2.12). Then,*

$$\{\tilde{D}_{\mathbf{m}}(a) \circ T_{\mathbf{m}}(f)\} \stackrel{\sigma}{\sim} a \otimes f.$$

Moreover, if a and f are real, then

$$\{\tilde{D}_{\mathbf{m}}(a) \circ T_{\mathbf{m}}(f)\} \stackrel{\lambda}{\sim} a \otimes f.$$

Proof. We decompose the Toeplitz matrix $T_{\mathbf{m}}(f)$ as

$$T_{\mathbf{m}}(f) = T_{\mathbf{m}}^D(f) + T_{\mathbf{m}}^L(f) + T_{\mathbf{m}}^U(f),$$

where $T_{\mathbf{m}}^D(f)$, $T_{\mathbf{m}}^L(f)$ and $T_{\mathbf{m}}^U(f)$ form the diagonal, lower and upper triangular matrix of $T_{\mathbf{m}}(f)$, respectively. Note that $T_{\mathbf{m}}^D(f)$, $T_{\mathbf{m}}^L(f)$ and $T_{\mathbf{m}}^U(f)$ are also d -level Toeplitz matrices associated with suitable trigonometric polynomials f^D , f^L and f^U such that $f = f^D + f^L + f^U$.

Then, one can verify that the matrix $\tilde{D}_{\mathbf{m}}(a) \circ T_{\mathbf{m}}(f)$ can be decomposed as

$$(2.13) \quad \tilde{D}_{\mathbf{m}}(a) \circ T_{\mathbf{m}}(f) = T_{\mathbf{m}}^L(f) D_{\mathbf{m}}(a) + D_{\mathbf{m}}(a) T_{\mathbf{m}}^D(f) + D_{\mathbf{m}}(a) T_{\mathbf{m}}^U(f),$$

where $D_{\mathbf{m}}(a)$ is the diagonal sampling matrix used in (2.12). Note that $\tilde{D}_{\mathbf{m}}(a)$ and $T_{\mathbf{m}}(f)$ are Hermitian because a and f are real, and so $\tilde{D}_{\mathbf{m}}(a) \circ T_{\mathbf{m}}(f)$ is

Hermitian as well. Hence, due to the decomposition (2.13), the spectral distribution of $\{\tilde{D}_m(a) \circ T_m(f)\}$ follows from Theorem 2.5. \square

Since many of the matrices appearing in Section 5 will be formed by a tensor-product of matrices defined in Section 4, we collect in the next lemma some basic results concerning tensor-products of matrices, see e.g. [5]. Without loss of clarity, we use the same symbol \otimes for both tensor-products of matrices and of functions.

Lemma 2.7. *Let $X \in \mathbb{C}^{m_1 \times m_1}$ and $Y \in \mathbb{C}^{m_2 \times m_2}$, then*

- a) $\text{rank}(X \otimes Y) = \text{rank}(X) \text{rank}(Y)$;
- b) $\|X \otimes Y\| = \|X\| \|Y\|$ and $\|X \otimes Y\|_1 = \|X\|_1 \|Y\|_1$.

The next result relates tensor-products and Toeplitz matrices. Given two (univariate) functions $f, h \in L_1([-\pi, \pi])$, we can construct the (bivariate) tensor-product function $f \otimes h$, which belongs to $L_1([-\pi, \pi]^2)$. Hence, we can consider the three families of Toeplitz matrices $\{T_{m_1}(f)\}$, $\{T_{m_2}(h)\}$ and $\{T_{m_1, m_2}(f \otimes h)\}$. A direct computation gives the following result.

Lemma 2.8. *Given $f, h \in L_1([-\pi, \pi])$, we have for all $m_1, m_2 \geq 1$,*

$$T_{m_1}(f) \otimes T_{m_2}(h) = T_{m_1, m_2}(f \otimes h).$$

3. CARDINAL B-SPLINES AND SYMBOLS

Let $\phi_{[p]}$ be the cardinal B-spline of degree p over the uniform knot sequence $\{0, 1, \dots, p+1\}$, which is defined recursively as follows [7]:

$$(3.1) \quad \phi_{[0]}(t) := \begin{cases} 1, & \text{if } t \in [0, 1), \\ 0, & \text{elsewhere,} \end{cases}$$

and

$$(3.2) \quad \phi_{[p]}(t) := \frac{t}{p} \phi_{[p-1]}(t) + \frac{p+1-t}{p} \phi_{[p-1]}(t-1), \quad p \geq 1.$$

As usual in the literature, we will refer to cardinal B-splines of degree p as the set of integer translates of $\phi_{[p]}$, that is $\{\phi_{[p]}(\cdot - k), k \in \mathbb{Z}\}$.

Denoting by \mathbb{P}_p the space of algebraic polynomials of degree less than or equal to p , it turns out that the cardinal B-spline $\phi_{[p]}$ belongs piecewisely to \mathbb{P}_p and it is globally of class C^{p-1} . It is well known that the cardinal B-spline possesses some fundamental properties. Some of them are briefly summarized below, see [7, 9].

- *Positivity:*

$$\phi_{[p]}(t) > 0, \quad t \in (0, p+1).$$

- *Minimal support:*

$$(3.3) \quad \phi_{[p]}(t) = 0, \quad t \notin [0, p+1].$$

- *Partition of unity:*

$$(3.4) \quad \sum_{k \in \mathbb{Z}} \phi_{[p]}(t - k) = 1.$$

- *Recurrence relation for derivatives:*

$$(3.5) \quad \phi_{[p]}^{(r)}(t) = \phi_{[p-1]}^{(r-1)}(t) - \phi_{[p-1]}^{(r-1)}(t-1).$$

We will denote by $\dot{\phi}_{[p]}(t)$ and $\ddot{\phi}_{[p]}(t)$ the first and second derivative of $\phi_{[p]}(t)$ with respect to its argument t . We recall from [9] that the Fourier transform of the cardinal B-spline $\phi_{[p]}$ and its derivative is given by

$$(3.6) \quad \widehat{\phi_{[p]}}(\theta) = \left(\widehat{\phi_{[0]}}(\theta) \right)^{p+1} = \left(\frac{1 - e^{-i\theta}}{i\theta} \right)^{p+1},$$

$$(3.7) \quad \widehat{\dot{\phi}_{[p]}}(\theta) = i\theta \widehat{\phi_{[p]}}(\theta) = (1 - e^{-i\theta}) \widehat{\phi_{[p-1]}}(\theta),$$

so that

$$(3.8) \quad \left| \widehat{\phi_{[p]}}(\theta) \right| = \left| \widehat{\phi_{[0]}}(\theta) \right|^{p+1} = \left(\frac{2 - 2\cos\theta}{\theta^2} \right)^{\frac{p+1}{2}} = \left| \frac{\sin(\theta/2)}{\theta/2} \right|^{p+1}.$$

We also note that for the symmetrized function $\phi_{[0]}^*(t) := \phi_{[0]}(t + 1/2)$, we have

$$(3.9) \quad \widehat{\phi_{[0]}^*}(\theta) = \frac{\sin(\theta/2)}{\theta/2}.$$

In addition, we recall the following results from [13, Lemmas 3 and 4].

Lemma 3.1. *Let $\phi_{[p]}$ be the cardinal B-spline as defined in (3.1)–(3.2), then*

$$\phi_{[p]}^{(r)}\left(\frac{p+1}{2} + t\right) = (-1)^r \phi_{[p]}^{(r)}\left(\frac{p+1}{2} - t\right).$$

Lemma 3.2. *Let $\phi_{[p]}$ be the cardinal B-spline as defined in (3.1)–(3.2), then*

$$\int_{\mathbb{R}} \phi_{[p_1]}^{(r)}(t) \phi_{[p_2]}^{(s)}(t+x) dt = (-1)^r \phi_{[p_1+p_2+1]}^{(r+s)}(p_1+1+x) = (-1)^s \phi_{[p_1+p_2+1]}^{(r+s)}(p_2+1-x).$$

We now analyze three functions associated to certain Toeplitz matrices of interest later on. More precisely, for $\theta \in [-\pi, \pi]$,

$$(3.10) \quad h_p(\theta) := \phi_{[p]}\left(\frac{p+1}{2}\right) + 2 \sum_{k=1}^{\lfloor p/2 \rfloor} \phi_{[p]}\left(\frac{p+1}{2} - k\right) \cos(k\theta), \quad p \geq 0,$$

$$(3.11) \quad g_p(\theta) := -2 \sum_{k=1}^{\lfloor p/2 \rfloor} \dot{\phi}_{[p]}\left(\frac{p+1}{2} - k\right) \sin(k\theta), \quad p \geq 2,$$

$$(3.12) \quad f_p(\theta) := -\ddot{\phi}_{[p]}\left(\frac{p+1}{2}\right) - 2 \sum_{k=1}^{\lfloor p/2 \rfloor} \ddot{\phi}_{[p]}\left(\frac{p+1}{2} - k\right) \cos(k\theta), \quad p \geq 2,$$

with the usual assumption that a sum is empty when the upper index is less than the lower one. Note that

$$\dot{\phi}_{[p]}\left(\frac{p+1}{2}\right) = 0,$$

and by using (3.1)–(3.2) and (3.5), it can be easily checked that

$$(3.13) \quad h_0(\theta) = h_1(\theta) = 1, \quad g_2(\theta) = g_3(\theta) = -\sin\theta, \quad f_2(\theta) = f_3(\theta) = 2 - 2\cos\theta.$$

Remark 3.3. The functions (3.10) and (3.12) have already been analyzed in [13] for odd degrees $p = 2q + 1$, $q \geq 1$. Note that f_p (resp. h_p) in [13] corresponds to f_{2q+1} (resp. h_{2q+1}) here. In addition, the parameter p in [13] corresponds to the parameter q here.

In the following lemmas we provide some properties of the functions h_p , g_p and f_p . They extend to any degree p the results obtained in [13, Lemma 6, Lemma 7 and Remark 2] for odd degree $p = 2q + 1$. The next lemma gives an alternative expression for h_p , g_p and f_p .

Lemma 3.4. *Let $p \geq 2$, and let h_p , g_p and f_p be the functions defined in (3.10)–(3.12). Then the following properties hold:*

a) $\forall \theta \in [-\pi, \pi],$

$$(3.14) \quad h_p(\theta) = \sum_{k \in \mathbb{Z}} \left(\widehat{\phi_{[0]}^*}(\theta + 2k\pi) \right)^{p+1} = \sum_{k \in \mathbb{Z}} \frac{(2 \sin(\theta/2 + k\pi))^{p+1}}{(\theta + 2k\pi)^{p+1}};$$

b) $\forall \theta \in [-\pi, \pi],$

$$(3.15) \quad g_p(\theta) = - \sum_{k \in \mathbb{Z}} \frac{(2 \sin(\theta/2 + k\pi))^{p+1}}{(\theta + 2k\pi)^p};$$

c) $\forall \theta \in [-\pi, \pi],$

$$(3.16) \quad f_p(\theta) = (2 - 2 \cos \theta) h_{p-2}(\theta),$$

and for $p \geq 4$,

$$(3.17) \quad f_p(\theta) = \sum_{k \in \mathbb{Z}} \frac{(2 \sin(\theta/2 + k\pi))^{p+1}}{(\theta + 2k\pi)^{p-1}}.$$

Proof. We first recall the Parseval identity for Fourier transforms, i.e.,

$$(3.18) \quad \int_{\mathbb{R}} \varphi(t) \overline{\psi(t)} dt = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\varphi}(\theta) \overline{\widehat{\psi}(\theta)} d\theta, \quad \varphi, \psi \in L_2(\mathbb{R}),$$

and the translation property of the Fourier transform, i.e.,

$$(3.19) \quad \widehat{\psi(\cdot + x)}(\theta) = \widehat{\psi}(\theta) e^{ix\theta}, \quad \psi \in L_1(\mathbb{R}), \quad x \in \mathbb{R}.$$

We differentiate the cases of odd and even degree p . We start with proving the relation (3.14) for $p = 2q$. From Lemma 3.2 we know that $\forall k \in \mathbb{Z}$,

$$(3.20) \quad \phi_{[p]} \left(\frac{p+1}{2} - k \right) = \phi_{[2q]} \left(q + \frac{1}{2} - k \right) = \int_{\mathbb{R}} \phi_{[q]}(t) \phi_{[q-1]} \left(t + k - \frac{1}{2} \right) dt.$$

In view of (3.18)–(3.19) and (3.6), for any $k \in \mathbb{Z}$ the expression in (3.20) is equal to

$$\begin{aligned} \int_{\mathbb{R}} \phi_{[q]}(t) \phi_{[q-1]} \left(t + k - \frac{1}{2} \right) dt &= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\phi_{[q]}}(\theta) \overline{\widehat{\phi_{[q-1]}}(\theta)} e^{-i(k-1/2)\theta} d\theta \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \left| \widehat{\phi_{[q-1]}}(\theta) \right|^2 \widehat{\phi_{[0]}}(\theta) e^{-i(k-1/2)\theta} d\theta \\ &= \frac{1}{2\pi} \sum_{l \in \mathbb{Z}} \int_{-\pi}^{\pi} \left| \widehat{\phi_{[q-1]}}(\theta + 2l\pi) \right|^2 \widehat{\phi_{[0]}}(\theta + 2l\pi) (-1)^l e^{-i(k-1/2)\theta} d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\sum_{l \in \mathbb{Z}} \left| \widehat{\phi_{[q-1]}}(\theta + 2l\pi) \right|^2 \widehat{\phi_{[0]}}(\theta + 2l\pi) (-1)^l e^{i\theta/2} \right] e^{-ik\theta} d\theta. \end{aligned}$$

Note that the last equality follows from the Dominated Convergence Theorem. Indeed, since $q \geq 1$, we obtain from (3.8) that

$$\begin{aligned} \int_{-\pi}^{\pi} \left| \sum_{l \in \mathbb{Z}} \left| \widehat{\phi_{[q-1]}}(\theta + 2l\pi) \right|^2 \widehat{\phi_{[0]}}(\theta + 2l\pi) (-1)^l e^{i\theta/2} e^{-ik\theta} \right| d\theta \\ \leq \int_{-\pi}^{\pi} \sum_{l \in \mathbb{Z}} \left(\frac{2 - 2 \cos \theta}{(\theta + 2l\pi)^2} \right)^{q+1/2} d\theta < \infty. \end{aligned}$$

We conclude that the values (3.20), i.e. the Fourier coefficients of h_{2q} in (3.10), are also the Fourier coefficients of the function

$$(3.21) \quad \sum_{l \in \mathbb{Z}} \left| \widehat{\phi_{[q-1]}}(\theta + 2l\pi) \right|^2 \widehat{\phi_{[0]}}(\theta + 2l\pi) (-1)^l e^{i\theta/2}.$$

Moreover, by using (3.6)–(3.9) we get

$$\widehat{\phi_{[0]}}(\theta + 2l\pi) (-1)^l e^{i\theta/2} = \frac{1 - e^{-i(\theta + 2l\pi)}}{i(\theta + 2l\pi)} e^{i(\theta + 2l\pi)/2} = \frac{\sin(\theta/2 + l\pi)}{\theta/2 + l\pi} = \widehat{\phi_{[0]}^*}(\theta + 2l\pi),$$

and it follows that h_{2q} is given by (3.14) for $q \geq 1$.

To prove the expression (3.15) of g_p for $p = 2q$, we follow an argument similar to the one in the proof of (3.14). Note that, by Lemma 3.2, we have $\forall k \in \mathbb{Z}$,

$$(3.22) \quad -\dot{\phi}_{[p]} \left(\frac{p+1}{2} - k \right) \frac{1}{i} = -\dot{\phi}_{[2q]} \left(q + \frac{1}{2} - k \right) \frac{1}{i} = \frac{1}{i} \int_{\mathbb{R}} \phi_{[q]}(t) \dot{\phi}_{[q-1]} \left(t + k - \frac{1}{2} \right) dt.$$

In view of (3.18)–(3.19) and (3.6)–(3.7), for any $k \in \mathbb{Z}$ the expression in (3.22) is equal to

$$\begin{aligned} \frac{1}{i} \int_{\mathbb{R}} \phi_{[q]}(t) \dot{\phi}_{[q-1]} \left(t + k - \frac{1}{2} \right) dt &= \frac{1}{2i\pi} \int_{\mathbb{R}} \widehat{\phi_{[q]}}(\theta) \overline{\widehat{\dot{\phi}_{[q-1]}}(\theta)} e^{-i(k-1/2)\theta} d\theta \\ &= -\frac{1}{2\pi} \int_{\mathbb{R}} \left| \widehat{\phi_{[q-1]}}(\theta) \right|^2 2 \sin(\theta/2) e^{-ik\theta} d\theta \\ &= -\frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\sum_{l \in \mathbb{Z}} \left| \widehat{\phi_{[q-1]}}(\theta + 2l\pi) \right|^2 2 \sin(\theta/2 + l\pi) \right] e^{-ik\theta} d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[-\sum_{l \in \mathbb{Z}} \frac{2(\sin(\theta/2 + l\pi))^{p+1}}{(\theta/2 + l\pi)^p} \right] e^{-ik\theta} d\theta. \end{aligned}$$

We conclude that the values (3.22), i.e. the Fourier coefficients of g_{2q} in (3.11), are also the Fourier coefficients of the function

$$-\sum_{l \in \mathbb{Z}} \frac{2(\sin(\theta/2 + l\pi))^{p+1}}{(\theta/2 + l\pi)^p} = -\sum_{l \in \mathbb{Z}} \frac{(2 \sin(\theta/2 + l\pi))^{p+1}}{(\theta + 2l\pi)^p}.$$

To prove the expression (3.16) of f_p for $p = 2q$, we follow again a similar argument as the one to prove (3.14). Note that, by Lemma 3.2, we have $\forall k \in \mathbb{Z}$,

$$(3.23) \quad -\ddot{\phi}_{[p]} \left(\frac{p+1}{2} - k \right) = -\ddot{\phi}_{[2q]} \left(q + \frac{1}{2} - k \right) = \int_{\mathbb{R}} \dot{\phi}_{[q]}(t) \dot{\phi}_{[q-1]} \left(t + k - \frac{1}{2} \right) dt.$$

In view of (3.18)–(3.19) and (3.6)–(3.7), for any $k \in \mathbb{Z}$ the expression in (3.23) is equal to

$$\begin{aligned} \int_{\mathbb{R}} \dot{\phi}_{[q]}(t) \dot{\phi}_{[q-1]}(t+k-1/2) dt &= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{\dot{\phi}_{[q]}}(\theta) \overline{\widehat{\dot{\phi}_{[q-1]}}(\theta)} e^{-i(k-1/2)\theta} d\theta \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \left| \widehat{\phi_{[q-2]}}(\theta) \right|^2 \widehat{\phi_{[0]}}(\theta) (2-2\cos\theta) e^{-i(k-1/2)\theta} d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\sum_{l \in \mathbb{Z}} \left| \widehat{\phi_{[q-2]}}(\theta+2l\pi) \right|^2 \widehat{\phi_{[0]}}(\theta+2l\pi) (-1)^l (2-2\cos\theta) e^{i\theta/2} \right] e^{-ik\theta} d\theta. \end{aligned}$$

We conclude that the values (3.23), i.e. the Fourier coefficients of f_{2q} in (3.11), are also the Fourier coefficients of the function

$$\sum_{l \in \mathbb{Z}} \left| \widehat{\phi_{[q-2]}}(\theta+2l\pi) \right|^2 \widehat{\phi_{[0]}}(\theta+2l\pi) (-1)^l (2-2\cos\theta) e^{i\theta/2}.$$

Hence, from (3.21) we obtain that $f_{2q}(\theta) = (2-2\cos\theta)h_{2(q-1)}(\theta)$ for $q \geq 2$. From (3.13) we see that the equality (3.16) holds for $q = 1$ as well. Moreover, (3.17) immediately follows from (3.14) and (3.16).

For odd degree $p = 2q + 1$, keeping Remark 3.3 in mind, we recall from [13, Lemma 7] that

$$h_{2q+1}(\theta) = \sum_{k \in \mathbb{Z}} \left| \widehat{\phi_{[q]}}(\theta+2k\pi) \right|^2, \quad q \geq 1.$$

In view of (3.8) and (3.9), we immediately obtain the relation (3.14). The equality (3.16) follows from [13, Remark 2] for $q \geq 2$ and from (3.13) for $q = 1$. Moreover, (3.17) is obtained by combining (3.14) and (3.16). The expression of g_{2q+1} can be shown by applying the same arguments as in the case of even degree. \square

To establish lower and upper bounds for h_p , g_p and f_p we need the following technical lemma.

Lemma 3.5. *Let $p \geq 2$, and let us consider the functions*

$$(3.24) \quad r_p(\theta) := \sum_{k \neq 0} \frac{(-1)^{k(p+1)}}{(\theta+2k\pi)^{p+1}}, \quad \tilde{r}_p(\theta) := - \sum_{k \neq 0} \frac{(-1)^{k(p+1)}}{(\theta+2k\pi)^p}, \quad \theta \in [-\pi, \pi].$$

Then, r_p and \tilde{r}_p are continuous functions over $[-\pi, \pi]$, and

$$(3.25) \quad 0 < r_p(\theta) \leq r_p(\pi) \leq \left(\frac{\pi^4}{48} - 1 \right) \frac{1}{\pi^{p+1}}, \quad \theta \in (0, \pi];$$

$$(3.26) \quad 0 < \tilde{r}_p(\theta) \leq \tilde{r}_p(\pi) = \frac{1}{\pi^p}, \quad \theta \in (0, \pi].$$

Proof. The functions r_p and \tilde{r}_p are continuous over $[-\pi, \pi]$ because the two series in (3.24) converge uniformly. We now derive an upper and lower bound for $r_p(\theta)$, $\theta \in [0, \pi]$. From (3.24) we obtain

$$(3.27) \quad r_p(\theta) = \sum_{k=1}^{\infty} \left[\frac{(-1)^{k(p+1)}}{(2k\pi+\theta)^{p+1}} + \frac{(-1)^{k(p+1)}}{(-2k\pi+\theta)^{p+1}} \right].$$

We differentiate the cases of odd and even degree. We first focus on the odd case $p = 2q + 1$, i.e., from (3.27)

$$r_{2q+1}(\theta) = \sum_{k=1}^{\infty} \left[\frac{1}{(2k\pi + \theta)^{2q+2}} + \frac{1}{(2k\pi - \theta)^{2q+2}} \right].$$

It is easy to see that $r_{2q+1}(\theta) > 0$ for $\theta \in [0, \pi]$. For $\rho > 1$, $k \geq 1$ and $\theta \in [0, \pi]$, one can check that

$$\frac{1}{(2k\pi + \theta)^\rho} + \frac{1}{(2k\pi - \theta)^\rho} \leq \frac{1}{(2k\pi + \pi)^\rho} + \frac{1}{(2k\pi - \pi)^\rho},$$

and then we obtain for $q \geq 1$,

$$\begin{aligned} r_{2q+1}(\theta) &\leq \sum_{k=1}^{\infty} \left[\frac{1}{(2k\pi + \pi)^{2q+2}} + \frac{1}{(2k\pi - \pi)^{2q+2}} \right] \\ &\leq \frac{1}{\pi^{2q+2}} \sum_{k=1}^{\infty} \left[\frac{1}{(2k+1)^4} + \frac{1}{(2k-1)^4} \right] = \frac{1}{\pi^{2q+2}} \left(\frac{\pi^4}{48} - 1 \right). \end{aligned}$$

We follow a similar argument for the even case $p = 2q$. In this case, from (3.27) we have

$$\begin{aligned} r_{2q}(\theta) &= \sum_{k=1}^{\infty} (-1)^k \left[\frac{1}{(2k\pi + \theta)^{2q+1}} - \frac{1}{(2k\pi - \theta)^{2q+1}} \right] = \sum_{l=1}^{\infty} \left[\frac{1}{(4l\pi + \theta)^{2q+1}} \right. \\ &\quad \left. - \frac{1}{(4l\pi - \theta)^{2q+1}} - \frac{1}{((4l-2)\pi + \theta)^{2q+1}} + \frac{1}{((4l-2)\pi - \theta)^{2q+1}} \right]. \end{aligned}$$

Let us define

$$s_\rho(\theta) := \frac{1}{(b + \theta)^\rho} - \frac{1}{(b - \theta)^\rho} + \frac{1}{(a - \theta)^\rho} - \frac{1}{(a + \theta)^\rho}.$$

If $\rho > 1$ and $0 \leq \theta \leq \pi < a < b$, then we have

$$s'_\rho(\theta) = -\frac{\rho}{(b + \theta)^{\rho+1}} - \frac{\rho}{(b - \theta)^{\rho+1}} + \frac{\rho}{(a - \theta)^{\rho+1}} + \frac{\rho}{(a + \theta)^{\rho+1}} > 0,$$

and thus s_ρ is a strictly increasing function, which implies that $s_\rho(\theta) > s_\rho(0) = 0$ for $\theta \in (0, \pi]$. As a consequence, we have $r_{2q}(\theta) > 0$ for $\theta \in (0, \pi]$. Moreover, for $q \geq 1$ and $\theta \in [0, \pi]$,

$$r_{2q}(\theta) \leq \sum_{l=1}^{\infty} \frac{1}{((4l-2)\pi - \theta)^{2q+1}} \leq \sum_{l=1}^{\infty} \frac{1}{((4l-2)\pi - \pi)^{2q+1}} \leq \frac{1}{\pi^{2q+1}} \sum_{l=1}^{\infty} \frac{1}{(4l-3)^3},$$

where

$$\sum_{l=1}^{\infty} \frac{1}{(4l-3)^3} < 1.02 < \frac{\pi^4}{48} - 1.$$

Hence, both in the odd and even case we obtain the bounds in (3.25).

We now derive an upper and lower bound for $\tilde{r}_p(\theta)$, $\theta \in [0, \pi]$. From (3.24) we have

$$\tilde{r}_p(\theta) = - \sum_{k=1}^{\infty} \left[\frac{(-1)^{k(p+1)}}{(2k\pi + \theta)^p} + \frac{(-1)^{k(p+1)}}{(-2k\pi + \theta)^p} \right].$$

We differentiate the cases of odd and even degree. We first focus on the odd case $p = 2q + 1$, i.e.,

$$\tilde{r}_{2q+1}(\theta) = \sum_{k=1}^{\infty} \frac{1}{(2k\pi - \theta)^{2q+1}} - \frac{1}{(2k\pi + \theta)^{2q+1}}.$$

We note that the function

$$\frac{1}{(a - \theta)^\rho} - \frac{1}{(a + \theta)^\rho}, \quad 0 \leq \theta \leq \pi < a, \quad \rho > 1,$$

is nonnegative and increasing. Then, for all $\theta \in (0, \pi]$ we have

$$0 < \tilde{r}_{2q+1}(\theta) \leq \tilde{r}_{2q+1}(\pi) = \frac{1}{\pi^{2q+1}} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2q+1}} - \frac{1}{(2k+1)^{2q+1}} = \frac{1}{\pi^{2q+1}},$$

which immediately gives (3.26).

Let us now consider the case $p = 2q$. We have

$$\tilde{r}_{2q}(\theta) = \sum_{k=1}^{\infty} \frac{1}{((4k-2)\pi + \theta)^{2q}} - \frac{1}{(4k\pi + \theta)^{2q}} + \frac{1}{((4k-2)\pi - \theta)^{2q}} - \frac{1}{(4k\pi - \theta)^{2q}}.$$

It can be easily checked that the function

$$\tilde{s}_\rho(\theta) := \frac{1}{(a + \theta)^\rho} - \frac{1}{(b + \theta)^\rho} + \frac{1}{(a - \theta)^\rho} - \frac{1}{(b - \theta)^\rho}, \quad 0 \leq \theta \leq \pi < a < b, \quad \rho > 1,$$

is positive, and $\tilde{s}'_\rho(\theta) = \rho s_{\rho+1}(\theta) > \tilde{s}'_\rho(0) = 0$ for $\theta \in (0, \pi]$. Therefore, \tilde{s}_ρ is increasing in $[0, \pi]$. As a consequence,

$$\begin{aligned} 0 < \tilde{r}_{2q}(\theta) \leq \tilde{r}_{2q}(\pi) &= \frac{1}{\pi^{2q}} \sum_{k=1}^{\infty} \frac{1}{(4k-1)^{2q}} - \frac{1}{(4k+1)^{2q}} + \frac{1}{(4k-3)^{2q}} - \frac{1}{(4k-1)^{2q}} \\ &= \frac{1}{\pi^{2q}} \sum_{k=1}^{\infty} \frac{1}{(4k-3)^{2q}} - \frac{1}{(4k+1)^{2q}} = \frac{1}{\pi^{2q}}. \end{aligned}$$

Thus we obtain (3.26) for the even case as well. \square

We now provide lower and upper bounds for h_p .

Lemma 3.6. *Let $p \geq 2$, and let h_p be the function defined in (3.10). Then the following properties hold:*

a) $\forall \theta \in [-\pi, \pi]$,

$$(3.28) \quad L_p(\theta) \leq h_p(\theta) \leq \min(1, U_p(\theta)),$$

where

$$(3.29) \quad L_p(\theta) := \left(\frac{2 - 2 \cos \theta}{\theta^2} \right)^{\frac{p+1}{2}},$$

$$(3.30) \quad U_p(\theta) := \left(\frac{2 - 2 \cos \theta}{\theta^2} \right)^{\frac{p+1}{2}} + \left(\frac{\pi^4}{48} - 1 \right) \left(\frac{2 - 2 \cos \theta}{\pi^2} \right)^{\frac{p+1}{2}};$$

b) $\max_{\theta \in [-\pi, \pi]} h_p(\theta) = h_p(0) = 1$;

c) let $m_{h_p} := \min_{\theta \in [-\pi, \pi]} h_p(\theta)$, then

$$(3.31) \quad m_{h_p} \geq \left(\frac{2}{\pi}\right)^{p+1} > 0;$$

d) the value $h_p(\pi)$ converges to 0 exponentially as $p \rightarrow \infty$, and in particular

$$(3.32) \quad h_p(\pi) \leq \frac{h_p(\pi)}{h_p(\frac{\pi}{2})} \leq 2^{\frac{1-p}{2}}.$$

Proof. First of all, we remark that h_p , L_p and U_p are symmetric around $\theta = 0$. Hence, it is sufficient to prove the various statements of the lemma for $\theta \in [0, \pi]$. We also recall that

$$\frac{\sin(\theta/2)}{\theta/2} = \left(\frac{2 - 2\cos\theta}{\theta^2}\right)^{1/2}, \quad \theta \in [-\pi, \pi].$$

Let us consider the first statement of the lemma. From (3.14) we obtain

$$h_p(\theta) = (\sin(\theta/2))^{p+1} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k(p+1)}}{(\theta/2 + k\pi)^{p+1}} = \left(\frac{\sin(\theta/2)}{\theta/2}\right)^{p+1} + (2\sin(\theta/2))^{p+1} r_p(\theta),$$

where r_p is defined in (3.24). Hence, from (3.25) we get

$$L_p(\theta) \leq h_p(\theta) \leq U_p(\theta).$$

We now focus on the second statement in the lemma. By using the local support property (3.3) and the partition of unity property (3.4) of cardinal B-splines, we obtain from (3.10) that

$$h_p(\theta) = \sum_{k \in \mathbb{Z}} \phi_{[p]} \left(\frac{p+1}{2} - k \right) e^{i(k\theta)} \leq \sum_{k \in \mathbb{Z}} \phi_{[p]} \left(\frac{p+1}{2} - k \right) |e^{i(k\theta)}| = 1.$$

In addition, it can be easily checked that $h_p(0) = 1$. This also completes the proof of the upper bound in (3.28).

To address the lower bound (3.31), we observe that $(2 - 2\cos\theta)/(\theta^2)$ is monotonically decreasing in $[0, \pi]$. As a consequence, we obtain that

$$(3.33) \quad L_p(\theta) \geq \left(\frac{2}{\pi}\right)^{p+1} > 0, \quad \theta \in [-\pi, \pi].$$

Finally, we focus on (3.32). For $p = 0, 1$ the statement follows from (3.13). Since $h_p(\theta) \leq 1$, it is sufficient to consider the ratio $h_p(\pi)/h_p(\pi/2)$ in order to prove (3.32) for the general case $p \geq 2$. From (3.14) we have

$$h_p\left(\frac{\pi}{2}\right) = \frac{2^{\frac{3(p+1)}{2}}}{\pi^{p+1}} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k(p+1)}}{(4k+1)^{p+1}}, \quad h_p(\pi) = \frac{2^{p+1}}{\pi^{p+1}} \sum_{k \in \mathbb{Z}} \frac{(-1)^{k(p+1)}}{(2k+1)^{p+1}}.$$

We differentiate the case of even and odd degree. We start with the even case $p = 2q$. Then,

$$h_{2q}\left(\frac{\pi}{2}\right) = \frac{2^{\frac{3(2q+1)}{2}}}{\pi^{2q+1}} \sum_{k \in \mathbb{Z}} \frac{(-1)^k}{(4k+1)^{2q+1}}, \quad h_{2q}(\pi) = \frac{2^{2q+1}}{\pi^{2q+1}} \sum_{k \in \mathbb{Z}} \frac{(-1)^k}{(2k+1)^{2q+1}}.$$

By splitting the latter sum into a sum over the even integers and a sum over the odd integers, we get

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \frac{(-1)^k}{(2k+1)^{2q+1}} &= \sum_{l \in \mathbb{Z}} \frac{1}{(4l+1)^{2q+1}} - \sum_{l \in \mathbb{Z}} \frac{1}{(4l+3)^{2q+1}} \\ &= \sum_{l \in \mathbb{Z}} \frac{1}{(4l+1)^{2q+1}} + \sum_{m \in \mathbb{Z}} \frac{1}{(4m+1)^{2q+1}} \\ &= \sum_{l \in \mathbb{Z}} \frac{2}{(4l+1)^{2q+1}} = 2(a_{2q} + b_{2q}), \end{aligned}$$

where

$$a_{2q} := \sum_{l \in \mathbb{Z}} \frac{1}{(8l+1)^{2q+1}}, \quad b_{2q} := \sum_{l \in \mathbb{Z}} \frac{1}{(8l+5)^{2q+1}}.$$

Hence,

$$h_{2q}\left(\frac{\pi}{2}\right) = \frac{2^{\frac{3(2q+1)}{2}}}{\pi^{2q+1}}(a_{2q} - b_{2q}), \quad h_{2q}(\pi) = \frac{2^{2q+1}}{\pi^{2q+1}}2(a_{2q} + b_{2q}).$$

It is easy to see that $b_{2q} < 0$. In addition, from (3.31) we know that $h_p(\theta) > 0$, so that $a_{2q} + b_{2q} > 0$, $a_{2q} - b_{2q} > 0$. Therefore, we obtain

$$\frac{h_{2q}(\pi)}{h_{2q}\left(\frac{\pi}{2}\right)} = \frac{2^{2q+2}(a_{2q} + b_{2q})}{2^{\frac{3(2q+1)}{2}}(a_{2q} - b_{2q})} \leq 2^{\frac{1-2q}{2}} = 2^{\frac{1-p}{2}}.$$

For odd degree $p = 2q + 1$, with a completely similar manipulation we obtain

$$\frac{h_{2q+1}(\pi)}{h_{2q+1}\left(\frac{\pi}{2}\right)} = 2^{\frac{1-p}{2}},$$

and so it follows that (3.32) holds again. \square

The next lemma is devoted to lower and upper bounds for g_p .

Lemma 3.7. *Let $p \geq 2$, and let g_p be the function defined in (3.11). Then the following properties hold:*

a) $\forall \theta \in [-\pi, \pi]$,

$$(3.34) \quad |2 \sin(\theta/2)|^{p+1} \left(\frac{1}{\theta^p} - \frac{1}{\pi^p} \right) \leq |g_p(\theta)| \leq |2 \sin(\theta/2)|^{p+1} \frac{1}{\theta^p};$$

b) the zeros of g_p are given by

$$(3.35) \quad g_p(-\pi) = g_p(0) = g_p(\pi) = 0.$$

Proof. We first remark that from (3.11) it follows that g_p is antisymmetric. Hence, it is sufficient to consider the interval $[0, \pi]$. From (3.15) and (3.24) we have

$$(3.36) \quad g_p(\theta) = - \sum_{k \in \mathbb{Z}} \frac{(2 \sin(\theta/2 + k\pi))^{p+1}}{(\theta + 2k\pi)^p} = -(2 \sin(\theta/2))^{p+1} \left[\frac{1}{\theta^p} - \tilde{r}_p(\theta) \right].$$

Then, (3.26) immediately gives (3.34) and (3.35). \square

In the following lemma we provide lower and upper bounds for f_p .

Lemma 3.8. *Let $p \geq 2$, and let f_p be the function defined in (3.12). Then the following properties hold:*

a) $\forall \theta \in [-\pi, \pi]$,

$$(3.37) \quad f_p(\theta) = 2 - 2 \cos \theta, \quad p = 2, 3,$$

and

$$(3.38) \quad (2 - 2 \cos \theta) L_{p-2}(\theta) \leq f_p(\theta) \leq (2 - 2 \cos \theta) \min(1, U_{p-2}(\theta)), \quad p \geq 4,$$

where L_p and U_p are defined in (3.29) and (3.30) respectively;

b) $\min_{\theta \in [-\pi, \pi]} f_p(\theta) = f_p(0) = 0$, and $\theta = 0$ is the unique zero of f_p in $[-\pi, \pi]$; this zero is of order two;

c) let $M_{f_p} := \max_{\theta \in [-\pi, \pi]} f_p(\theta)$, then

$$(3.39) \quad M_{f_p} \leq \min \left(4, \frac{16}{p+1} + \left(\frac{\pi^4}{12} - 4 \right) \left(\frac{2}{\pi} \right)^{p-1} \right),$$

so that $M_{f_p} \rightarrow 0$ as $p \rightarrow \infty$;

d) the ratio $f_p(\pi)/M_{f_p}$ converges to 0 exponentially as $p \rightarrow \infty$, and in particular

$$(3.40) \quad \frac{f_p(\pi)}{M_{f_p}} \leq \frac{f_p(\pi)}{f_p(\frac{\pi}{2})} \leq 2^{\frac{5-p}{2}}.$$

Proof. The first statement of the lemma immediately follows from (3.13), (3.16) and (3.28).

The relations (3.37)–(3.38) and the lower bound (3.33) imply that $f_p(\theta) \geq 0$ in $[-\pi, \pi]$ and that it has a unique zero at $\theta = 0$ in $[-\pi, \pi]$. Moreover, from (3.16) we obtain

$$\begin{aligned} f_p'(\theta) &= 2(\sin \theta) h_{p-2}(\theta) + (2 - 2 \cos \theta) h_{p-2}'(\theta), \\ f_p''(\theta) &= 2(\cos \theta) h_{p-2}(\theta) + 4(\sin \theta) h_{p-2}'(\theta) + (2 - 2 \cos \theta) h_{p-2}''(\theta). \end{aligned}$$

By using (3.13) and the second statement of Lemma 3.6, we get $f_p'(0) = 0$ and $f_p''(0) = 2$. This proves that f_p has a zero of order two at $\theta = 0$ and completes the second statement of the lemma.

From (3.37)–(3.38) it is also easy to see that $M_{f_p} \leq 4$. Now, we derive the upper bound (3.39) for M_{f_p} in the third statement of the lemma. To this end, we use the inequalities

$$2 - 2 \cos \theta \leq \theta^2 - \frac{\theta^4}{18} \leq \theta^2, \quad \forall \theta \in [-\pi, \pi].$$

It follows

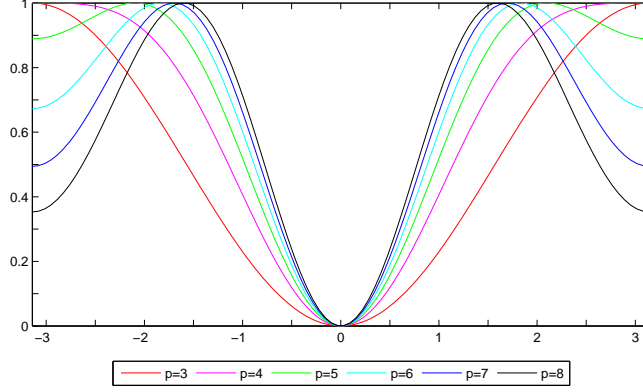
$$(2 - 2 \cos \theta) \left(\frac{2 - 2 \cos \theta}{\theta^2} \right)^{\frac{p-1}{2}} \leq \theta^2 \left(1 - \frac{\theta^2}{18} \right)^{\frac{p-1}{2}}, \quad \forall \theta \in [-\pi, \pi].$$

Let ρ be a positive real number. If $\frac{18}{\rho+1} \leq \pi^2$, then the maximum of $\theta^2 \left(1 - \frac{\theta^2}{18} \right)^\rho$ over $[-\pi, \pi]$ is located at $\theta^2 = \frac{18}{\rho+1}$ and its value is given by

$$\frac{18}{\rho+1} \left(1 - \frac{1}{\rho+1} \right)^\rho \leq \frac{8}{\rho+1}.$$

Therefore, when $p \geq 3$, we have

$$(3.41) \quad (2 - 2 \cos \theta) \left(\frac{2 - 2 \cos \theta}{\theta^2} \right)^{\frac{p-1}{2}} \leq \frac{16}{p+1}, \quad \forall \theta \in [-\pi, \pi].$$

FIGURE 1. Graph of f_p/M_{f_p} for $p = 3, \dots, 8$.

Moreover, $\forall \theta \in [-\pi, \pi]$,

$$(3.42) \quad (2 - 2 \cos \theta) \left(\frac{\pi^4}{48} - 1 \right) \left(\frac{2 - 2 \cos \theta}{\pi^2} \right)^{\frac{p-1}{2}} \leq 4 \left(\frac{\pi^4}{48} - 1 \right) \left(\frac{2}{\pi} \right)^{p-1}.$$

From (3.30) and (3.38), the inequalities (3.41)–(3.42) imply that, for $p \geq 4$,

$$(3.43) \quad M_{f_p} \leq \frac{16}{p+1} + \left(\frac{\pi^4}{12} - 4 \right) \left(\frac{2}{\pi} \right)^{p-1}.$$

In addition, (3.43) holds for $p = 2$ and $p = 3$ too, because of (3.37) and $M_{f_2} = M_{f_3} = 4$.

To conclude the proof, we notice that the inequalities in (3.40) follow from (3.32) taking into account that $f_p(\theta) = (2 - 2 \cos \theta)h_{p-2}(\theta)$. \square

Figure 1 shows the graph of f_p normalized by its maximum M_{f_p} for $p = 3, \dots, 8$. The ratio $f_p(\pi)/M_{f_p}$ decreases exponentially to zero as $p \rightarrow \infty$, see (3.40). From a numerical viewpoint, we can say that, for large p , the function f_p/M_{f_p} possesses two zeros over $[0, \pi]$: one in $\theta = 0$ and the other in $\theta = \pi$.

Finally, we provide an important relation between the functions h_p , g_p and f_p .

Lemma 3.9. *For all $\theta \in [-\pi, \pi] \setminus \{0\}$, we have*

$$(3.44) \quad f_p(\theta)h_p(\theta) - (g_p(\theta))^2 > 0.$$

Proof. From (3.28), (3.38) and (3.34) we have

$$f_p(\theta)h_p(\theta) - (g_p(\theta))^2 \geq \frac{(2 \sin(\theta/2))^{2p+2}}{(\theta^p)^2} - (g_p(\theta))^2 \geq 0.$$

Moreover, since $\tilde{r}_p(\theta)$ is strictly positive if $\theta \neq 0$ (see (3.26)), from (3.36) we obtain the complete statement of the lemma. \square

4. THE 1D SETTING

Without loss of generality we can assume $\Omega = (0, 1)$. Then we focus on the problem

$$(4.1) \quad \begin{cases} -\kappa(x)u''(x) + \beta(x)u'(x) + \gamma(x)u(x) = f(x), & 0 < x < 1, \\ u(0) = 0, \quad u(1) = 0. \end{cases}$$

Fix $p \geq 2$, $n \geq 2$ and let $\mathcal{V}_n^{[p]}$ be the space of splines of degree p (or order $p+1$) defined over the knot sequence

$$(4.2) \quad t_1 = \dots = t_{p+1} = 0 < t_{p+2} < \dots < t_{p+n} < 1 = t_{p+n+1} = \dots = t_{2p+n+1},$$

where

$$(4.3) \quad t_{p+i+1} := \frac{i}{n}, \quad \forall i = 0, \dots, n,$$

and the extreme knots have multiplicity $p+1$. More precisely,

$$\mathcal{V}_n^{[p]} := \{s \in C^{p-1}([0, 1]) : s|_{[t_{p+i+1}, t_{p+i+2}]} \in \mathbb{P}_p, \forall i = 0, 1, \dots, n-1\}.$$

Let $\mathcal{W}_n^{[p]}$ be the subspace of $\mathcal{V}_n^{[p]}$ formed by the spline functions vanishing at the boundary of $[0, 1]$, i.e.,

$$\mathcal{W}_n^{[p]} := \{s \in \mathcal{V}_n^{[p]} : s(0) = s(1) = 0\}.$$

We recall that $\dim \mathcal{V}_n^{[p]} = n + p$ and $\dim \mathcal{W}_n^{[p]} = n + p - 2$.

The space $\mathcal{V}_n^{[p]}$ is spanned by the B-spline basis $\{N_{i,[p]}(x), i = 1, \dots, n+p\}$ defined as follows (see [7]). Using the convention that a fraction with zero denominator is zero, define the function $N_{i,[k]} : [0, 1] \rightarrow \mathbb{R}$ for every (k, i) such that $0 \leq k \leq p$, $1 \leq i \leq (n+p) + p - k$:

$$N_{i,[0]}(x) := \begin{cases} 1, & \text{if } x \in [t_i, t_{i+1}), \\ 0, & \text{elsewhere,} \end{cases}$$

and, if $k > 0$,

$$(4.4) \quad N_{i,[k]}(x) := \frac{x - t_i}{t_{i+k} - t_i} N_{i,[k-1]}(x) + \frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}} N_{i+1,[k-1]}(x).$$

We point out that the derivative of a B-spline $N_{i,[p]}(x)$ is given by (see [7])

$$(4.5) \quad N'_{i,[p]}(x) = p \left(\frac{N_{i,[p-1]}(x)}{t_{i+p} - t_i} - \frac{N_{i+1,[p-1]}(x)}{t_{i+p+1} - t_{i+1}} \right).$$

The set $\{N_{i,[p]} : i = 1, \dots, n+p\}$ is a basis of $\mathcal{V}_n^{[p]}$, called the B-spline basis of $\mathcal{V}_n^{[p]}$. Moreover, recalling that [7]

$$N_{i,[p]}(0) = N_{i,[p]}(1) = 0, \quad \forall i = 2, \dots, n+p-1,$$

we deduce that $\{N_{i,[p]} : i = 2, \dots, n+p-1\}$ is a basis of $\mathcal{W}_n^{[p]}$, called the B-spline basis of $\mathcal{W}_n^{[p]}$.

A particular interesting set of collocation points for the B-spline space are the Greville abscissae, see [2]. They are defined as

$$(4.6) \quad \xi_{i,[p]} := \frac{t_{i+1} + t_{i+2} + \dots + t_{i+p}}{p}, \quad i = 2, \dots, n+p-1.$$

In the isogeometric collocation approach, the approximation $u_{\mathcal{W}} = \sum_{j=1}^{n+p-2} u_j \varphi_j$ of the solution of (4.1) is obtained by solving the linear system (2.4) where the basis functions are given by (2.7) with

$$(4.7) \quad \hat{\varphi}_j(\hat{x}) = N_{j+1,[p]}(\hat{x}), \quad j = 1, \dots, n+p-2,$$

and

$$(4.8) \quad \hat{\tau}_j = \xi_{j+1,[p]}, \quad j = 1, \dots, n+p-2.$$

The matrix A in (2.4) based on (2.7)–(2.8) and (4.7)–(4.8) is the object of our interest and, from now onwards, will be denoted by $A_n^{[p]}$ in order to emphasize its dependence on n and p . For the sake of simplicity, we first analyze the matrix $A_n^{[p]}$ when G is the identity map, before considering the general case related to a non-trivial geometry map (see Section 4.3).

4.1. Construction of the matrices $A_n^{[p]}$ without geometry map. When G is the identity map, our matrix $A = A_n^{[p]}$ in (2.4) is equal to

$$(4.9) \quad A = \left[-\kappa(\xi_{i+1,[p]}) N_{j+1,[p]}''(\xi_{i+1,[p]}) + \beta(\xi_{i+1,[p]}) N_{j+1,[p]}'(\xi_{i+1,[p]}) + \gamma(\xi_{i+1,[p]}) N_{j+1,[p]}(\xi_{i+1,[p]}) \right]_{i,j=1}^{n+p-2},$$

and $\mathbf{f} = [\mathbf{f}(\xi_{i+1,[p]})]_{i=1}^{n+p-2}$. The elements of this matrix can be computed by using the recurrence relation (4.4) and by iterating the derivative formula (4.5).

As mentioned in our previous paper [13], the ‘central’ basis functions $N_{j,[p]}(x)$, $j = p+1, \dots, n$, defined on the knot sequence (4.2)–(4.3) are cardinal B-splines, and we have

$$(4.10) \quad N_{j,[p]}(x) = \phi_{[p]}(nx - j + p + 1), \quad j = p+1, \dots, n.$$

Due to the compact support of the B-spline basis, the matrix $A_n^{[p]}$ has at most a $(p+1)$ -band structure. We note that

$$N_{j,[p]}'(x) = n \dot{\phi}_{[p]}(nx - j + p + 1), \quad j = p+1, \dots, n,$$

and

$$N_{j,[p]}''(x) = n^2 \ddot{\phi}_{[p]}(nx - j + p + 1), \quad j = p+1, \dots, n.$$

In addition, the interior Greville abscissae, given by (4.6) for $i = p+1, \dots, n$, simplify to

$$(4.11) \quad \xi_{i,[p]} = \frac{i}{n} - \frac{p+1}{2n}, \quad i = p+1, \dots, n,$$

or, equivalently,

$$n\xi_{i,[p]} + p + 1 = i + \frac{p+1}{2}, \quad i = p+1, \dots, n.$$

Let us denote by $D_n^{[p]}(a)$ the diagonal matrix containing the samples of the function a at the Greville abscissae, i.e.,

$$D_n^{[p]}(a) := \text{diag}_{j=1, \dots, n+p-2} (a(\xi_{j+1,[p]})).$$

Then, we can consider the following split of $A_n^{[p]}$,

$$(4.12) \quad A_n^{[p]} = n^2 D_n^{[p]}(\kappa) K_n^{[p]} + n D_n^{[p]}(\beta) H_n^{[p]} + D_n^{[p]}(\gamma) M_n^{[p]},$$

according to the diffusion, advection and reaction terms, respectively. More precisely,

$$(4.13) \quad n^2 K_n^{[p]} := \left[-N''_{j+1,[p]}(\xi_{i+1,[p]}) \right]_{i,j=1}^{n+p-2},$$

$$(4.14) \quad n H_n^{[p]} := \left[N'_{j+1,[p]}(\xi_{i+1,[p]}) \right]_{i,j=1}^{n+p-2},$$

$$(4.15) \quad M_n^{[p]} := \left[N_{j+1,[p]}(\xi_{i+1,[p]}) \right]_{i,j=1}^{n+p-2}.$$

We now focus on the central submatrix of $A_n^{[p]}$ that is determined only by the cardinal B-splines in (4.10) and by the interior Greville abscissae (4.11), i.e., the submatrix $\left[(A_n^{[p]})_{ij} \right]_{i,j=p}^{n-1}$. We assume $n \geq p+1$ to be sure that this submatrix exists. Then, the submatrices $\left[(K_n^{[p]})_{ij} \right]_{i,j=p}^{n-1}$, $\left[(H_n^{[p]})_{ij} \right]_{i,j=p}^{n-1}$ and $\left[(M_n^{[p]})_{ij} \right]_{i,j=p}^{n-1}$ are given by

$$\begin{aligned} (K_n^{[p]})_{ij} &= -\ddot{\phi}_{[p]} \left(\frac{p+1}{2} + i - j \right), \\ (H_n^{[p]})_{ij} &= \dot{\phi}_{[p]} \left(\frac{p+1}{2} + i - j \right), \\ (M_n^{[p]})_{ij} &= \phi_{[p]} \left(\frac{p+1}{2} + i - j \right), \end{aligned}$$

for $i, j = p, \dots, n-1$. From Lemma 3.1 it follows that the above central submatrices of $K_n^{[p]}$ and $M_n^{[p]}$ are symmetric, whereas the above central submatrix of $H_n^{[p]}$ is skew-symmetric. We note that the coefficients depend only on the difference $i-j$, and so all the above submatrices are Toeplitz matrices of size $(n-p) \times (n-p)$. Moreover, because of the local support property (3.3), for every fixed $i = p, \dots, n-1$, the i -th row of $K_n^{[p]}$, $H_n^{[p]}$ and $M_n^{[p]}$ contains nonzero entries only for the column index ranging from $j = i - \lceil (p-1)/2 \rceil, \dots, i + \lceil (p-1)/2 \rceil$.

We conclude this section by giving a formal definition of what we call ‘central rows’ of $A_n^{[p]}$. Assuming for the moment that $n \gg p$, we look for indices $i \in \{p, \dots, n-1\}$ such that

$$(4.16) \quad (A_n^{[p]})_{i,1} = \dots = (A_n^{[p]})_{i,p-1} = 0 = (A_n^{[p]})_{i,n} = \dots = (A_n^{[p]})_{i,n+p-2}.$$

To find indices $i \in \{p, \dots, n-1\}$ such that (4.16) is satisfied, we fix $i \in \{p, \dots, n-1\}$ and we recall that $\text{supp}(N_{2,[p]}) \subseteq \dots \subseteq \text{supp}(N_{p,[p]}) = [0, \frac{p}{n}]$ and $[1 - \frac{p}{n}, 1] = \text{supp}(N_{n+1,[p]}) \supseteq \dots \supseteq \text{supp}(N_{n+p-1,[p]})$. From (4.9) and (4.11) it follows that condition (4.16) is implied by

$$\begin{aligned} \xi_{i+1,[p]} \notin \left[0, \frac{p}{n} \right] \cup \left[1 - \frac{p}{n}, 1 \right] &\Leftrightarrow \frac{p}{n} \leq \xi_{i+1,[p]} \leq 1 - \frac{p}{n} \\ &\Leftrightarrow \frac{p}{n} \leq \frac{i+1}{n} - \frac{p+1}{2n} \leq 1 - \frac{p}{n} \\ &\Leftrightarrow \frac{3p-1}{2} \leq i \leq n+p-1 - \frac{3p-1}{2}. \end{aligned}$$

Note that $p \leq \frac{3p-1}{2}$ and $n+p-1 - \frac{3p-1}{2} \leq n-1$. In view of these results, ‘central rows’ of $A_n^{[p]}$ are those rows of $A_n^{[p]}$ with index ranging from $i = \lceil (3p-1)/2 \rceil, \dots, n+$

$p-1 - \lceil (3p-1)/2 \rceil$. Consequently, a condition to ensure that $A_n^{[p]}$ has at least one central row is $n \geq p^*$, where

$$(4.17) \quad p^* := \begin{cases} 2p, & \text{if } p \text{ is odd,} \\ 2p+1, & \text{if } p \text{ is even.} \end{cases}$$

4.2. Spectral analysis, symbol and distribution without geometry map.

We will now study, for a fixed $p \geq 2$, the spectral distribution of the sequence formed by the scaled matrices:

$$(4.18) \quad \frac{1}{n^2} A_n^{[p]} = D_n^{[p]}(\kappa) K_n^{[p]} + \frac{1}{n} D_n^{[p]}(\beta) H_n^{[p]} + \frac{1}{n^2} D_n^{[p]}(\gamma) M_n^{[p]}.$$

For every $n \geq p^*$, we decompose the matrix $K_n^{[p]}$ into

$$(4.19) \quad K_n^{[p]} = T_{n+p-2}(f_p) + R_n^{[p]},$$

where f_p is defined in (3.12) and $T_{n+p-2}(f_p)$ is the corresponding (one-level) Toeplitz matrix defined in (2.9). By the construction of our diffusion matrix we know that $R_n^{[p]} := K_n^{[p]} - T_{n+p-2}(f_p)$ is a low-rank correction term. Indeed, $R_n^{[p]}$ has at most $2\lceil (3p-1)/2 \rceil - 2$ non-zero rows (all the central rows are null) and hence

$$(4.20) \quad \text{rank}(R_n^{[p]}) \leq 2\lceil (3p-1)/2 \rceil - 2, \quad \forall n \geq p^*.$$

Similarly, we decompose the matrices $H_n^{[p]}$ and $M_n^{[p]}$ into

$$(4.21) \quad H_n^{[p]} = i T_{n+p-2}(g_p) + Q_n^{[p]}, \quad M_n^{[p]} = T_{n+p-2}(h_p) + S_n^{[p]},$$

where g_p and h_p are defined in (3.11) and (3.10) respectively. Moreover, $Q_n^{[p]} := H_n^{[p]} - i T_{n+p-2}(g_p)$ and $S_n^{[p]} := M_n^{[p]} - T_{n+p-2}(h_p)$ are again low-rank correction terms analogous to $R_n^{[p]}$.

The spectral analysis can then be performed in a way similar to the Galerkin case considered in [13]. We first provide upper bounds for the 2-norm of the matrices $K_n^{[p]}$, $H_n^{[p]}$, $M_n^{[p]}$ in the next lemma.

Lemma 4.1. *For every $p \geq 2$ and every $n \geq 2$, we have*

$$\|M_n^{[p]}\| \leq \sqrt{\frac{3p}{2}}, \quad \|H_n^{[p]}\| \leq p\sqrt{3p}, \quad \|K_n^{[p]}\| \leq 2p(p-1)\sqrt{3p}.$$

Proof. The 2-norm of any square matrix X can be bounded by

$$(4.22) \quad \|X\| \leq \sqrt{\|X\|_\infty \|X^T\|_\infty},$$

see e.g. [6, p. 121]. Hence, we now look for bounds of the infinity norm of the matrices $K_n^{[p]}$, $H_n^{[p]}$, $M_n^{[p]}$ and their transposes.

We first bound the infinity norm of $K_n^{[p]}$, $H_n^{[p]}$, $M_n^{[p]}$. From (4.15), the positivity property and the partition of unity property of B-splines, we obtain

$$\|M_n^{[p]}\|_\infty = \max_{i=1, \dots, n+p-2} \sum_{j=1}^{n+p-2} N_{j+1, [p]}(\xi_{i+1, [p]}) \leq 1.$$

Similarly, from (4.14), (4.5), the partition of unity property of B-splines, and by taking into account that the sequence of knots (4.2)–(4.3) implies that $t_{i+p+1} -$

$t_{i+1} \geq \frac{1}{n}$ for all $i = 1, \dots, n+p-1$, we have

$$\begin{aligned} \|nH_n^{[p]}\|_\infty &= \max_{i=1, \dots, n+p-2} \sum_{j=1}^{n+p-2} \left| N'_{j+1, [p]}(\xi_{i+1, [p]}) \right| \\ &\leq \max_{i=1, \dots, n+p-2} p \sum_{j=1}^{n+p-2} \left(\frac{N_{j+1, [p-1]}(\xi_{i+1, [p]})}{t_{j+p+1} - t_{j+1}} + \frac{N_{j+2, [p-1]}(\xi_{i+1, [p]})}{t_{j+p+2} - t_{j+2}} \right) \leq 2pn. \end{aligned}$$

By using similar arguments and by iterating (4.5), we obtain from (4.13) that

$$\|n^2 K_n^{[p]}\|_\infty = \max_{i=1, \dots, n+p-2} \sum_{j=1}^{n+p-2} \left| N''_{j+1, [p]}(\xi_{i+1, [p]}) \right| \leq 4p(p-1)n^2.$$

We now bound the infinity norm of the transposes of $K_n^{[p]}$, $H_n^{[p]}$, $M_n^{[p]}$. The number of Greville abscissae in the interior of the support of any B-spline is at most $\frac{3p}{2}$. In combination with the positivity property and the partition of unity property of B-splines, we obtain

$$\|(M_n^{[p]})^T\|_\infty = \max_{j=1, \dots, n+p-2} \sum_{i=1}^{n+p-2} N_{j+1, [p]}(\xi_{i+1, [p]}) \leq \frac{3p}{2}.$$

From (4.5), and by exploiting again the properties of the B-splines, we have

$$|N'_{j+1, [p]}(\xi_{i+1, [p]})| \leq p \left(\frac{N_{j+1, [p-1]}(\xi_{i+1, [p]})}{t_{j+p+1} - t_{j+1}} + \frac{N_{j+2, [p-1]}(\xi_{i+1, [p]})}{t_{j+p+2} - t_{j+2}} \right) \leq pn,$$

and so

$$\|(nH_n^{[p]})^T\|_\infty = \max_{j=1, \dots, n+p-2} \sum_{i=1}^{n+p-2} \left| N'_{j+1, [p]}(\xi_{i+1, [p]}) \right| \leq pn \frac{3p}{2}.$$

Similarly, by iterating (4.5), we obtain

$$\|(n^2 K_n^{[p]})^T\|_\infty = \max_{i=1, \dots, n+p-2} \sum_{j=1}^{n+p-2} \left| N''_{j+1, [p]}(\xi_{i+1, [p]}) \right| \leq 2p(p-1)n^2 \frac{3p}{2}.$$

The proof is completed by combining the above bounds as given by (4.22). \square

The next theorem provides us the symbol of the sequence $\{\frac{1}{n^2} A_n^{[p]}\}$. For this, we need the concept of modulus of continuity, which is defined for any (multivariate) function $s : \overline{\Omega} \rightarrow \mathbb{R}$, and $\delta > 0$ as

$$\omega(s, \delta) := \sup_{\mathbf{x}, \mathbf{y} \in \overline{\Omega}, \|\mathbf{x} - \mathbf{y}\| \leq \delta} |s(\mathbf{x}) - s(\mathbf{y})|.$$

In the following, we denote by C a generic constant independent of n .

Theorem 4.2. *Let $p \geq 2$. Then, the sequence of normalized collocation matrices $\{\frac{1}{n^2} A_n^{[p]}\}$ as in (4.9) is distributed like the function $\kappa \otimes f_p$ in the sense of the eigenvalues, i.e., $\forall F \in C_c(\mathbb{C}, \mathbb{C})$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n+p-2} \sum_{j=1}^{n+p-2} F \left(\lambda_j \left(\frac{1}{n^2} A_n^{[p]} \right) \right) = \frac{1}{2\pi} \int_0^1 \int_{-\pi}^\pi F(\kappa(x) f_p(\theta)) d\theta dx.$$

Furthermore, if

$$(4.23) \quad \omega\left(\kappa, \frac{1}{n}\right) \leq \frac{C}{n},$$

then $\{\frac{1}{n^2}A_n^{[p]}\}$ is strongly clustered at the range $[0, M_{\kappa, f_p}]$, where

$$M_{\kappa, f_p} := \max_{x \in [0, 1], \theta \in [-\pi, \pi]} \kappa(x) f_p(\theta).$$

Proof. Let $\tilde{D}_{n+p-2}(\kappa)$ be the symmetric matrix defined in (2.12) which is constructed from the diagonal sampling matrix $D_{n+p-2}(\kappa)$. Then, we decompose the expression of $\frac{1}{n^2}A_n^{[p]}$ in (4.18) as follows:

$$\frac{1}{n^2}A_n^{[p]} = \tilde{D}_{n+p-2}(\kappa) \circ T_{n+p-2}(f_p) + E_n^{[p]} + F_n^{[p]} + L_n^{[p]},$$

where

$$\begin{aligned} E_n^{[p]} &:= \frac{1}{n} D_n^{[p]}(\beta) H_n^{[p]} + \frac{1}{n^2} D_n^{[p]}(\gamma) M_n^{[p]}, \\ F_n^{[p]} &:= D_n^{[p]}(\kappa) K_n^{[p]} - D_n^{[p]}(\kappa) T_{n+p-2}(f_p). \end{aligned}$$

and

$$L_n^{[p]} := D_n^{[p]}(\kappa) T_{n+p-2}(f_p) - \tilde{D}_{n+p-2}(\kappa) \circ T_{n+p-2}(f_p).$$

We now prove that the hypotheses of Theorem 2.3 are satisfied with $Z_n = \frac{1}{n^2}A_n^{[p]}$, $X_n = \tilde{D}_{n+p-2}(\kappa) \circ T_{n+p-2}(f_p)$ and $Y_n = E_n^{[p]} + F_n^{[p]} + L_n^{[p]}$.

We know that $T_{n+p-2}(f_p)$ is symmetric. Then, from Corollary 2.6 it follows that $\{\tilde{D}_{n+p-2}(\kappa) \circ T_{n+p-2}(f_p)\} \stackrel{\lambda}{\sim} \kappa \otimes f_p$. From (2.10) we have $\|T_{n+p-2}(f_p)\| \leq M_{f_p}$, where $M_{f_p} := \max_{\theta \in [-\pi, \pi]} f_p(\theta)$. Moreover, the band structure of $T_{n+p-2}(f_p)$ implies that $\|T_{n+p-2}(f_p)\|_\infty \leq C_T$, and

$$\|\tilde{D}_{n+p-2}(\kappa) \circ T_{n+p-2}(f_p)\| \leq \|\tilde{D}_{n+p-2}(\kappa) \circ T_{n+p-2}(f_p)\|_\infty \leq M_\kappa C_T,$$

where $M_\kappa := \max_{x \in [0, 1]} \kappa(x)$ and C_T are constants independent of n .

From Lemma 4.1 we have

$$\|E_n^{[p]}\| \leq M_{|\beta|} \frac{p\sqrt{3p}}{n} + M_\gamma \frac{1}{n^2} \sqrt{\frac{3p}{2}},$$

and

$$\|E_n^{[p]}\|_1 \leq (n+p-2) \|E_n^{[p]}\| \leq (n+p-2) \left(M_{|\beta|} \frac{p\sqrt{3p}}{n} + M_\gamma \frac{1}{n^2} \sqrt{\frac{3p}{2}} \right),$$

where $M_{|\beta|} := \max_{x \in [0, 1]} |\beta(x)|$ and $M_\gamma := \max_{x \in [0, 1]} \gamma(x)$. Therefore, $\|E_n^{[p]}\|$ and $\|E_n^{[p]}\|_1$ are bounded above by a constant independent of n .

From Lemma 4.1 we have

$$\|F_n^{[p]}\| \leq M_\kappa (\|K_n^{[p]}\| + \|T_{n+p-2}(f_p)\|) \leq M_\kappa (2p(p-1)\sqrt{3p} + M_{f_p}),$$

and since $R_n^{[p]} = K_n^{[p]} - T_{n+p-2}(f_p)$ and $\text{rank}(R_n^{[p]}) \leq 2\lceil(3p-1)/2\rceil - 2$, see (4.19)–(4.20), we obtain

$$\begin{aligned} \|F_n^{[p]}\|_1 &= \|D_n^{[p]}(\kappa) R_n^{[p]}\|_1 \leq (2\lceil(3p-1)/2\rceil - 2) \|D_n^{[p]}(\kappa) R_n^{[p]}\| \\ &\leq (2\lceil(3p-1)/2\rceil - 2) M_\kappa (2p(p-1)\sqrt{3p} + M_{f_p}). \end{aligned}$$

Therefore, $\|F_n^{[p]}\|$ and $\|F_n^{[p]}\|_1$ are bounded above by a constant independent of n .

We now focus on the matrix $L_n^{[p]}$, which shares the same band structure as the Toeplitz matrix $T_{n+p-2}(f_p)$. This band has a width of at most $p + 1$. We can express any non-zero element of $L_n^{[p]}$ at row i and column j by

$$(L_n^{[p]})_{i,j} = (T_{n+p-2}(f_p))_{i,j} \left(\kappa(\xi_{i+1,[p]}) - \kappa\left(\frac{\min(i,j)-1}{n+p-2}\right) \right),$$

where $0 \leq |i-j| \leq \lfloor \frac{p}{2} \rfloor$. Since

$$(4.24) \quad \left| \xi_{i+1,[p]} - \frac{\min(i,j)-1}{n+p-2} \right| \leq \frac{C}{n},$$

we have

$$|(L_n^{[p]})_{i,j}| \leq |(T_{n+p-2}(f_p))_{i,j}| \omega\left(\kappa, \frac{C}{n}\right).$$

It follows that both $\|L_n^{[p]}\|_\infty$ and $\|(L_n^{[p]})^T\|_\infty$ are bounded above by the expression $\omega\left(\kappa, \frac{C}{n}\right) \|T_{n+p-2}(f_p)\|_\infty$, and so by (4.22) we have

$$\|L_n^{[p]}\| \leq \omega\left(\kappa, \frac{C}{n}\right) \|T_{n+p-2}(f_p)\|_\infty \leq C_T \omega\left(\kappa, \frac{C}{n}\right).$$

This means that

$$(4.25) \quad \frac{\|L_n^{[p]}\|_1}{n+p-2} \leq \|L_n^{[p]}\| \leq C_T \omega\left(\kappa, \frac{C}{n}\right).$$

Since κ is continuous over the interval $[0, 1]$, we obtain that $\lim_{n \rightarrow \infty} \omega\left(\kappa, \frac{C}{n}\right) = 0$, and this implies

$$\lim_{n \rightarrow \infty} \frac{\|L_n^{[p]}\|_1}{n+p-2} = 0.$$

Summarizing,

$$\|E_n^{[p]} + F_n^{[p]} + L_n^{[p]}\| \leq \|E_n^{[p]}\| + \|F_n^{[p]}\| + \|L_n^{[p]}\| \leq C,$$

which is an upper bound independent of n , and

$$\lim_{n \rightarrow \infty} \frac{\|E_n^{[p]} + F_n^{[p]} + L_n^{[p]}\|_1}{n+p-2} = 0.$$

Hence, all the hypotheses of Theorem 2.3 are satisfied, and we conclude that $\{\frac{1}{n^2} A_n^{[p]}\} \stackrel{\lambda}{\sim} \kappa \otimes f_p$.

In addition, if (4.23) holds, then (4.25) implies that $\|L_n^{[p]}\|_1$ is bounded above by a constant independent of n , and all the hypotheses of Theorem 2.4 are satisfied as well. This implies that $\{\frac{1}{n^2} A_n^{[p]}\}$ is strongly clustered at $[0, M_{\kappa, f_p}]$. \square

Remark 4.3. If κ is a C^1 -continuous function over $[0, 1]$, then the condition (4.23) is satisfied, so that in this case the sequence $\{\frac{1}{n^2} A_n^{[p]}\}$ is strongly clustered at the range $[0, M_{\kappa, f_p}]$.

Remark 4.4. From Lemma 3.8 we know that the symbol $\kappa(x)f_p(\theta)$ of the sequence of scaled collocation matrices will have a zero at $\theta = 0$, and moreover a numerical zero at $\theta = \pi$ for large p .

Remark 4.5. Referring to [13, Theorem 12], in the case of constant coefficients, we note that the symbol derived for the scaled stiffness matrix in the Galerkin formulation with B-splines of degree q is the same as for the scaled matrix in the collocation formulation with B-splines of odd degree $2q + 1$.

4.3. Construction and symbol of the matrices $A_n^{[p]}$ with a geometry map.

The general case of isogeometric collocation methods with a non-trivial geometry map can be easily addressed with the aid of the results from the previous subsection. Indeed, the geometry map only invokes a change of variable, and leads to a new formulation of the problem (4.1) with different coefficients. Given a geometry map $G : [0, 1] \rightarrow [0, 1]$, our model problem becomes

$$(4.26) \quad \begin{cases} -\frac{\kappa(G(\hat{x}))}{(G'(\hat{x}))^2} u''(\hat{x}) + \left(\frac{\kappa(G(\hat{x}))G''(\hat{x})}{(G'(\hat{x}))^3} + \frac{\beta(G(\hat{x}))}{G'(\hat{x})} \right) u'(\hat{x}) + \gamma(G(\hat{x}))u(\hat{x}) = f(G(\hat{x})), \\ u(0) = 0, \quad u(1) = 0, \end{cases}$$

with $0 < \hat{x} < 1$. Hence, our matrix $A = A_n^{[p]}$ in (2.4) with (4.7)–(4.8) is equal to

$$(4.27) \quad A_n^{[p]} = \left[-\frac{\kappa(G(\xi_{i+1,[p]}))}{(G'(\xi_{i+1,[p]}))^2} N_{j+1,[p]}''(\xi_{i+1,[p]}) + \left(\frac{\kappa(G(\xi_{i+1,[p]}))G''(\xi_{i+1,[p]})}{(G'(\xi_{i+1,[p]}))^3} + \frac{\beta(G(\xi_{i+1,[p]}))}{G'(\xi_{i+1,[p]})} \right) N_{j+1,[p]}'(\xi_{i+1,[p]}) + \gamma(G(\xi_{i+1,[p]})) N_{j+1,[p]}(\xi_{i+1,[p]}) \right]_{i,j=1}^{n+p-2},$$

and $\mathbf{f} = [f(G(\xi_{i+1,[p]}))]_{i=1}^{n+p-2}$.

The next theorem shows explicitly the influence of the geometry map on the spectral distribution of the scaled matrices $\{\frac{1}{n^2} A_n^{[p]}\}$.

Theorem 4.6. *Let $p \geq 2$. Let $G : [0, 1] \rightarrow [0, 1]$ such that $G \in C^1([0, 1])$, $0 < G'(\hat{x})$ for all $\hat{x} \in [0, 1]$ and G'' is bounded. Then, the sequence of normalized collocation matrices $\{\frac{1}{n^2} A_n^{[p]}\}$ as in (4.27) is distributed like the function $\frac{\kappa(G)}{(G')^2} \otimes f_p$ in the sense of the eigenvalues. Furthermore, if*

$$(4.28) \quad \omega \left(\frac{\kappa(G)}{(G')^2}, \frac{1}{n} \right) \leq \frac{C}{n},$$

then $\{\frac{1}{n^2} A_n^{[p]}\}$ is strongly clustered at the range $[0, M_{\frac{\kappa(G)}{(G')^2}, f_p}]$, where

$$M_{\frac{\kappa(G)}{(G')^2}, f_p} := \max_{\hat{x} \in [0, 1], \theta \in [-\pi, \pi]} \frac{\kappa(G(\hat{x}))}{(G'(\hat{x}))^2} f_p(\theta).$$

Proof. This follows immediately from Theorem 4.2 by comparing the matrices (4.27) and (4.9). \square

Remark 4.7. The considered geometry map G in Theorem 4.6 can be given in any representation and is not confined to the B-spline form (2.6) as prescribed by the IgA paradigm.

5. THE 2D SETTING

We now consider our model problem (2.1) on a two-dimensional domain Ω , namely

$$(5.1) \quad \begin{cases} -\mathbf{1}(K(\mathbf{x}) \circ Pu(\mathbf{x}))\mathbf{1}^T + \boldsymbol{\beta}(\mathbf{x}) \cdot \nabla u(\mathbf{x}) + \gamma(\mathbf{x})u(\mathbf{x}) = f(\mathbf{x}), & \forall \mathbf{x} \in \Omega, \\ u(\mathbf{x}) = 0, & \forall \mathbf{x} \in \partial\Omega, \end{cases}$$

where $K = \begin{bmatrix} \kappa_{1,1} & \kappa_{1,2} \\ \kappa_{2,1} & \kappa_{2,2} \end{bmatrix}$ and $\boldsymbol{\beta} = [\beta_1 \ \beta_2]^T$.

We will approximate the solution of problem (5.1) in the space of smooth tensor-product splines that we now describe. We consider two univariate B-spline bases and sets of Greville abscissae as defined in Section 4 (for the x and y directions):

- the B-spline basis $\{N_{i,[p_1]}(x), \ i = 1, \dots, n_1 + p_1\}$ and the corresponding Greville abscissae $\xi_{i,[p_1]}$ over the knot sequence $\{s_i, \ i = 1, \dots, 2p_1 + n_1 + 1\}$, similar to (4.2)–(4.3);
- the B-spline basis $\{N_{i,[p_2]}(y), \ i = 1, \dots, n_2 + p_2\}$ and the corresponding Greville abscissae $\xi_{i,[p_2]}$ over the knot sequence $\{t_i, \ i = 1, \dots, 2p_2 + n_2 + 1\}$, similar to (4.2)–(4.3).

The bivariate tensor-product B-spline basis $\{N_{i,j,[p_1,p_2]}, \ i = 1, \dots, n_1 + p_1, \ j = 1, \dots, n_2 + p_2\}$ is given by

$$N_{i,j,[p_1,p_2]}(x, y) := (N_{i,[p_1]} \otimes N_{j,[p_2]})(x, y) = N_{i,[p_1]}(x)N_{j,[p_2]}(y),$$

and the corresponding Greville abscissa is given by

$$\boldsymbol{\xi}_{i,j,[p_1,p_2]} := (\xi_{i,[p_1]}, \xi_{j,[p_2]}).$$

We choose the space $\mathcal{W}_{n_1,n_2}^{[p_1,p_2]}$ as approximation space, where

$$(5.2) \quad \mathcal{W}_{n_1,n_2}^{[p_1,p_2]} := \langle N_{i,j,[p_1,p_2]} : \ i = 2, \dots, n_1 + p_1 - 1, \ j = 2, \dots, n_2 + p_2 - 1 \rangle,$$

and we consider the elements of the basis (5.2) ordered as follows:

$$(5.3) \quad \hat{\varphi}_{(n_1+p_1-2)(j-1)+i} = N_{i+1,j+1,[p_1,p_2]},$$

with $i = 1, \dots, n_1 + p_1 - 2, \ j = 1, \dots, n_2 + p_2 - 2$. We follow the same rule to order the Greville abscissae:

$$(5.4) \quad \hat{\tau}_{(n_1+p_1-2)(j-1)+i} = \boldsymbol{\xi}_{i+1,j+1,[p_1,p_2]},$$

with $i = 1, \dots, n_1 + p_1 - 2, \ j = 1, \dots, n_2 + p_2 - 2$.

The corresponding collocation matrix A in (2.4) based on (2.7)–(2.8) and (5.3)–(5.4) is the object of our interest and, from now onwards, will be denoted by $A_{n_1,n_2}^{[p_1,p_2]}$ in order to emphasize its dependence on n_1, n_2 and p_1, p_2 . We first consider our model problem on the reference domain $\Omega = (0, 1)^2$ and then we will address the case of a general two-dimensional physical domain Ω , parametrized by a non-trivial geometry map from $\hat{\Omega} = [0, 1]^2$.

5.1. Construction and symbol of the matrices $A_{n_1,n_2}^{[p_1,p_2]}$ without geometry map. When using the identity geometry map, the matrix $A = A_{n_1,n_2}^{[p_1,p_2]}$ in (2.4) has the following expression:

$$(5.5) \quad A_{n_1,n_2}^{[p_1,p_2]} = \left[-\mathbf{1}(K(\hat{\tau}_i) \circ P\hat{\varphi}_j(\hat{\tau}_i))\mathbf{1}^T + \boldsymbol{\beta}(\hat{\tau}_i) \cdot \nabla \hat{\varphi}_j(\hat{\tau}_i) + \gamma(\hat{\tau}_i)\hat{\varphi}_j(\hat{\tau}_i) \right]_{i,j=1}^{(n_1+p_1-2)(n_2+p_2-2)}.$$

Let us denote by $D_{n_1, n_2}^{[p_1, p_2]}(a)$ the two-level diagonal matrix containing the samples of the function a at the Greville abscissae, i.e.,

$$D_{n_1, n_2}^{[p_1, p_2]}(a) := \text{diag}_{j=1, \dots, n_2+p_2-2} \left(\text{diag}_{i=1, \dots, n_1+p_1-2} (a(\xi_{i+1, j+1, [p_1, p_2]})) \right).$$

By using the univariate collocation matrices in Section 4.1, we obtain that the matrix in (5.5) can be written as

$$(5.6) \quad \begin{aligned} A_{n_1, n_2}^{[p_1, p_2]} = & n_1^2 D_{n_1, n_2}^{[p_1, p_2]}(\kappa_{1,1}) \widehat{K}_{n_1, n_2}^{[p_1, p_2]} + n_1 n_2 D_{n_1, n_2}^{[p_1, p_2]}(\kappa_{1,2} + \kappa_{2,1}) \overline{K}_{n_1, n_2}^{[p_1, p_2]} \\ & + n_2^2 D_{n_1, n_2}^{[p_1, p_2]}(\kappa_{2,2}) \widetilde{K}_{n_1, n_2}^{[p_1, p_2]} + n_1 D_{n_1, n_2}^{[p_1, p_2]}(\beta_1) \widehat{H}_{n_1, n_2}^{[p_1, p_2]} \\ & + n_2 D_{n_1, n_2}^{[p_1, p_2]}(\beta_2) \widetilde{H}_{n_1, n_2}^{[p_1, p_2]} + D_{n_1, n_2}^{[p_1, p_2]}(\gamma) M_{n_1, n_2}^{[p_1, p_2]}, \end{aligned}$$

where

$$\widehat{K}_{n_1, n_2}^{[p_1, p_2]} := M_{n_2}^{[p_2]} \otimes K_{n_1}^{[p_1]}, \quad \overline{K}_{n_1, n_2}^{[p_1, p_2]} := -H_{n_2}^{[p_2]} \otimes H_{n_1}^{[p_1]}, \quad \widetilde{K}_{n_1, n_2}^{[p_1, p_2]} := K_{n_2}^{[p_2]} \otimes M_{n_1}^{[p_1]},$$

$$\widehat{H}_{n_1, n_2}^{[p_1, p_2]} := M_{n_2}^{[p_2]} \otimes H_{n_1}^{[p_1]}, \quad \widetilde{H}_{n_1, n_2}^{[p_1, p_2]} := H_{n_2}^{[p_2]} \otimes M_{n_1}^{[p_1]}, \quad M_{n_1, n_2}^{[p_1, p_2]} := M_{n_2}^{[p_2]} \otimes M_{n_1}^{[p_1]}.$$

We will now study, for fixed $p_1, p_2 \geq 2$, the spectral distribution of the sequence of matrices (5.6) under the additional mild assumption that $\frac{n_1}{n} =: \nu_1$ and $\frac{n_2}{n} =: \nu_2$ are constants as $n \rightarrow \infty$.² Let $\mathbf{n} := (\nu_2 n + p_2 - 2, \nu_1 n + p_1 - 2)$. For every n such that $n_1 \geq p_1^*$ and $n_2 \geq p_2^*$, see (4.17), we decompose the matrices $\widehat{K}_{n_1, n_2}^{[p_1, p_2]}$, $\overline{K}_{n_1, n_2}^{[p_1, p_2]}$ and $\widetilde{K}_{n_1, n_2}^{[p_1, p_2]}$ into

$$\begin{aligned} \widehat{B}_{n_1, n_2}^{[p_1, p_2]} &:= T_{n_2+p_2-2}(h_{p_2}) \otimes T_{n_1+p_1-2}(f_{p_1}) = T_{\mathbf{n}}(h_{p_2} \otimes f_{p_1}), \\ \overline{B}_{n_1, n_2}^{[p_1, p_2]} &:= -iT_{n_2+p_2-2}(g_{p_2}) \otimes iT_{n_1+p_1-2}(g_{p_1}) = T_{\mathbf{n}}(g_{p_2} \otimes g_{p_1}), \\ \widetilde{B}_{n_1, n_2}^{[p_1, p_2]} &:= T_{n_2+p_2-2}(f_{p_2}) \otimes T_{n_1+p_1-2}(h_{p_1}) = T_{\mathbf{n}}(f_{p_2} \otimes h_{p_1}), \end{aligned}$$

and

$$\begin{aligned} \widehat{R}_{n_1, n_2}^{[p_1, p_2]} &:= \widehat{K}_{n_1, n_2}^{[p_1, p_2]} - \widehat{B}_{n_1, n_2}^{[p_1, p_2]}, \\ \overline{R}_{n_1, n_2}^{[p_1, p_2]} &:= \overline{K}_{n_1, n_2}^{[p_1, p_2]} - \overline{B}_{n_1, n_2}^{[p_1, p_2]}, \\ \widetilde{R}_{n_1, n_2}^{[p_1, p_2]} &:= \widetilde{K}_{n_1, n_2}^{[p_1, p_2]} - \widetilde{B}_{n_1, n_2}^{[p_1, p_2]}. \end{aligned}$$

We note that

$$\widehat{R}_{n_1, n_2}^{[p_1, p_2]} = T_{n_2+p_2-2}(h_{p_2}) \otimes R_{n_1}^{[p_1]} + S_{n_2}^{[p_2]} \otimes T_{n_1+p_1-2}(f_{p_1}) + S_{n_2}^{[p_2]} \otimes R_{n_1}^{[p_1]},$$

²In this way, $A_{n_1, n_2}^{[p_1, p_2]}$ is really a sequence of matrices, since only n is a free parameter. The relations $n_1 = \nu_1 n$ and $n_2 = \nu_2 n$ must be kept in mind while reading this section.

where the matrices $R_m^{[p]}$, $S_m^{[p]}$ were introduced in (4.19) and (4.21). Since they are matrices of low rank independent of m , see (4.20), we obtain from Lemma 2.7 that

$$\begin{aligned}
(5.7) \quad & \text{rank}(\widehat{R}_{n_1, n_2}^{[p_1, p_2]}) \\
& \leq \text{rank}(T_{n_2+p_2-2}(h_{p_2}) \otimes R_{n_1}^{[p_1]}) + \text{rank}(S_{n_2}^{[p_2]} \otimes T_{n_1+p_1-2}(f_{p_1})) \\
& \quad + \text{rank}(S_{n_2}^{[p_2]} \otimes R_{n_1}^{[p_1]}) \\
& = \text{rank}(T_{n_2+p_2-2}(h_{p_2})) \text{rank}(R_{n_1}^{[p_1]}) + \text{rank}(S_{n_2}^{[p_2]}) \text{rank}(T_{n_1+p_1-2}(f_{p_1})) \\
& \quad + \text{rank}(S_{n_2}^{[p_2]}) \text{rank}(R_{n_1}^{[p_1]}) \\
& \leq (\nu_2 n + p_2 - 2)2(2p_1 - 1) + 2(2p_2 - 1)(\nu_1 n + p_1 - 2) + 2(2p_2 - 1)2(2p_1 - 1) \\
& = o((\nu_1 n + p_1 - 2)(\nu_2 n + p_2 - 2)), \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

Similar relations hold for $\text{rank}(\overline{R}_{n_1, n_2}^{[p_1, p_2]})$ and $\text{rank}(\widetilde{R}_{n_1, n_2}^{[p_1, p_2]})$.

The spectral distribution of the scaled matrix $\frac{1}{n^2} A_{n_1, n_2}^{[p_1, p_2]}$ is given by the following theorem.

Theorem 5.1. *The sequence of matrices $\{\frac{1}{n^2} A_{n_1, n_2}^{[p_1, p_2]}\}_n$ (with $n_1 = \nu_1 n$ and $n_2 = \nu_2 n$) is distributed like the function*

$$(\nu_1)^2 \kappa_{1,1} \otimes h_{p_2} \otimes f_{p_1} + \nu_1 \nu_2 (\kappa_{1,2} + \kappa_{2,1}) \otimes g_{p_2} \otimes g_{p_1} + (\nu_2)^2 \kappa_{2,2} \otimes f_{p_2} \otimes h_{p_1},$$

in the sense of the eigenvalues, where the functions f_p , g_p and h_p are defined in (3.12), (3.11) and (3.10), respectively.

Proof. We just provide an outline of the proof, since it combines the arguments used in the Proof of Theorem 4.2 and in the Proof of [13, Theorem 18]. In the following, C will denote a generic constant independent of n . We decompose the expression of $\frac{1}{n^2} A_{n_1, n_2}^{[p_1, p_2]}$ as follows

$$\begin{aligned}
\frac{1}{n^2} A_{n_1, n_2}^{[p_1, p_2]} &= (\nu_1)^2 \widetilde{D}_{\mathbf{n}}(\kappa_{1,1}) \circ \widehat{B}_{n_1, n_2}^{[p_1, p_2]} + \nu_1 \nu_2 \widetilde{D}_{\mathbf{n}}(\kappa_{1,2} + \kappa_{2,1}) \circ \overline{B}_{n_1, n_2}^{[p_1, p_2]} \\
&\quad + (\nu_2)^2 \widetilde{D}_{\mathbf{n}}(\kappa_{2,2}) \circ \widetilde{B}_{n_1, n_2}^{[p_1, p_2]} + L_{n_1, n_2}^{[p_1, p_2]} + F_{n_1, n_2}^{[p_1, p_2]} + E_{n_1, n_2}^{[p_1, p_2]},
\end{aligned}$$

where

$$\begin{aligned}
L_{n_1, n_2}^{[p_1, p_2]} &:= (\nu_1)^2 \widehat{L}_{n_1, n_2}^{[p_1, p_2]} + \nu_1 \nu_2 \overline{L}_{n_1, n_2}^{[p_1, p_2]} + (\nu_2)^2 \widetilde{L}_{n_1, n_2}^{[p_1, p_2]}, \\
\widehat{L}_{n_1, n_2}^{[p_1, p_2]} &:= D_{n_1, n_2}^{[p_1, p_2]}(\kappa_{1,1}) \widehat{B}_{n_1, n_2}^{[p_1, p_2]} - \widetilde{D}_{\mathbf{n}}(\kappa_{1,1}) \circ \widehat{B}_{n_1, n_2}^{[p_1, p_2]}, \\
\overline{L}_{n_1, n_2}^{[p_1, p_2]} &:= D_{n_1, n_2}^{[p_1, p_2]}(\kappa_{1,2} + \kappa_{2,1}) \overline{B}_{n_1, n_2}^{[p_1, p_2]} - \widetilde{D}_{\mathbf{n}}(\kappa_{1,2} + \kappa_{2,1}) \circ \overline{B}_{n_1, n_2}^{[p_1, p_2]}, \\
\widetilde{L}_{n_1, n_2}^{[p_1, p_2]} &:= D_{n_1, n_2}^{[p_1, p_2]}(\kappa_{2,2}) \widetilde{B}_{n_1, n_2}^{[p_1, p_2]} - \widetilde{D}_{\mathbf{n}}(\kappa_{2,2}) \circ \widetilde{B}_{n_1, n_2}^{[p_1, p_2]},
\end{aligned}$$

and

$$\begin{aligned}
F_{n_1, n_2}^{[p_1, p_2]} &:= (\nu_1)^2 D_{n_1, n_2}^{[p_1, p_2]}(\kappa_{1,1}) \widehat{R}_{n_1, n_2}^{[p_1, p_2]} + \nu_1 \nu_2 D_{n_1, n_2}^{[p_1, p_2]}(\kappa_{1,2} + \kappa_{2,1}) \overline{R}_{n_1, n_2}^{[p_1, p_2]} \\
&\quad + (\nu_2)^2 D_{n_1, n_2}^{[p_1, p_2]}(\kappa_{2,2}) \widetilde{R}_{n_1, n_2}^{[p_1, p_2]}, \\
E_{n_1, n_2}^{[p_1, p_2]} &:= \frac{\nu_1}{n} D_{n_1, n_2}^{[p_1, p_2]}(\beta_1) \widehat{H}_{n_1, n_2}^{[p_1, p_2]} + \frac{\nu_2}{n} D_{n_1, n_2}^{[p_1, p_2]}(\beta_2) \widetilde{H}_{n_1, n_2}^{[p_1, p_2]} + \frac{1}{n^2} D_{n_1, n_2}^{[p_1, p_2]}(\gamma) M_{n_1, n_2}^{[p_1, p_2]}.
\end{aligned}$$

We now prove that the hypotheses of Theorem 2.3 are satisfied with

$$\begin{aligned} Z_n &= \frac{1}{n^2} A_{n_1, n_2}^{[p_1, p_2]}, \\ X_n &= (\nu_1)^2 \tilde{D}_n(\kappa_{1,1}) \circ \hat{B}_{n_1, n_2}^{[p_1, p_2]} + \nu_1 \nu_2 \tilde{D}_n(\kappa_{1,2} + \kappa_{2,1}) \circ \bar{B}_{n_1, n_2}^{[p_1, p_2]} \\ &\quad + (\nu_2)^2 \tilde{D}_n(\kappa_{2,2}) \circ \tilde{B}_{n_1, n_2}^{[p_1, p_2]}, \\ Y_n &= L_{n_1, n_2}^{[p_1, p_2]} + F_{n_1, n_2}^{[p_1, p_2]} + E_{n_1, n_2}^{[p_1, p_2]}. \end{aligned}$$

We first remark that $\hat{B}_{n_1, n_2}^{[p_1, p_2]} = T_n(h_{p_2} \otimes f_{p_1})$, $\tilde{B}_{n_1, n_2}^{[p_1, p_2]} = T_n(f_{p_2} \otimes h_{p_1})$ and $\bar{B}_{n_1, n_2}^{[p_1, p_2]} = T_n(g_{p_2} \otimes g_{p_1})$ are symmetric. Then, from Corollary 2.6 it follows that

$$X_n \stackrel{\lambda}{\sim} (\nu_1)^2 \kappa_{1,1} \otimes h_{p_2} \otimes f_{p_1} + \nu_1 \nu_2 (\kappa_{1,2} + \kappa_{2,1}) \otimes g_{p_2} \otimes g_{p_1} + (\nu_2)^2 \kappa_{2,2} \otimes f_{p_2} \otimes h_{p_1}.$$

Moreover, the 2-level band structure of X_n and the continuity of $\kappa_{i,j}$, $i, j = 1, 2$ imply that

$$\|X_n\| \leq \|X_n\|_\infty \leq C.$$

Now we analyze the behavior of the perturbation matrix Y_n . First we address $E_{n_1, n_2}^{[p_1, p_2]}$. From Lemmas 2.7 and 4.1, we obtain

$$\|\hat{H}_{n_1, n_2}^{[p_1, p_2]}\|, \|\tilde{H}_{n_1, n_2}^{[p_1, p_2]}\|, \|M_{n_1, n_2}^{[p_1, p_2]}\| \leq C.$$

Therefore, by the continuity of β and γ we have

$$\|E_{n_1, n_2}^{[p_1, p_2]}\| \leq O\left(\frac{1}{n}\right),$$

and

$$\lim_{n \rightarrow \infty} \frac{\|E_{n_1, n_2}^{[p_1, p_2]}\|_1}{(\nu_1 n + p_1 - 2)(\nu_2 n + p_2 - 2)} \leq \lim_{n \rightarrow \infty} \|E_{n_1, n_2}^{[p_1, p_2]}\| = 0.$$

Note that $(\nu_1 n + p_1 - 2)(\nu_2 n + p_2 - 2)$ is the dimension of the matrix $A_{n_1, n_2}^{[p_1, p_2]}$.

Let us now consider $F_{n_1, n_2}^{[p_1, p_2]}$. We have

$$\begin{aligned} \|\hat{R}_{n_1, n_2}^{[p_1, p_2]}\| &= \|\hat{K}_{n_1, n_2}^{[p_1, p_2]} - \hat{B}_{n_1, n_2}^{[p_1, p_2]}\| \\ &\leq \|M_{n_2}^{[p_2]}\| \|K_{n_1}^{[p_1]}\| + \|T_{n_2+p_2-2}(h_{p_2})\| \|T_{n_1+p_1-2}(f_{p_1})\|, \end{aligned}$$

and similar bounds hold for $\|\bar{R}_{n_1, n_2}^{[p_1, p_2]}\|$ and $\|\tilde{R}_{n_1, n_2}^{[p_1, p_2]}\|$. Then, from Lemma 4.1, from the properties of Toeplitz and diagonal matrices, see e.g. (2.10), and from the continuity of $\kappa_{i,j}$, $i, j = 1, 2$ we obtain

$$\|F_{n_1, n_2}^{[p_1, p_2]}\| \leq C,$$

and by the low rank (5.7) of $\hat{R}_{n_1, n_2}^{[p_1, p_2]}$, $\bar{R}_{n_1, n_2}^{[p_1, p_2]}$ and $\tilde{R}_{n_1, n_2}^{[p_1, p_2]}$ we have

$$\begin{aligned} &\lim_{n \rightarrow \infty} \frac{\|F_{n_1, n_2}^{[p_1, p_2]}\|_1}{(\nu_1 n + p_1 - 2)(\nu_2 n + p_2 - 2)} \\ &\leq \lim_{n \rightarrow \infty} \frac{\text{rank}(F_{n_1, n_2}^{[p_1, p_2]})}{(\nu_1 n + p_1 - 2)(\nu_2 n + p_2 - 2)} \|F_{n_1, n_2}^{[p_1, p_2]}\| = 0. \end{aligned}$$

Finally, let us focus on $\hat{L}_{n_1, n_2}^{[p_1, p_2]}$ which shares the same sparsity structure as the 2-level Toeplitz matrix $\hat{B}_{n_1, n_2}^{[p_1, p_2]} = T_n(h_{p_2} \otimes f_{p_1})$. Since $\hat{B}_{n_1, n_2}^{[p_1, p_2]}$ is the tensor-product

of two banded matrices, we have $\|\widehat{B}_{n_1, n_2}^{[p_1, p_2]}\|_\infty \leq C$ and $\|(\widehat{B}_{n_1, n_2}^{[p_1, p_2]})^T\|_\infty \leq C$. Hence, from (4.22) and from the continuity of $\kappa_{1,1}$ we deduce

$$\|\widetilde{D}_{\mathbf{n}}(\kappa_{1,1}) \circ \widehat{B}_{n_1, n_2}^{[p_1, p_2]}\| \leq M_{\kappa_{1,1}} C,$$

where $M_{\kappa_{1,1}} := \max_{(x,y) \in \overline{\Omega}} \kappa_{1,1}(x,y)$. We can express any non-zero element of $\widehat{L}_{n_1, n_2}^{[p_1, p_2]}$ at row i and column j by

$$(5.8) \quad (\widehat{L}_{n_1, n_2}^{[p_1, p_2]})_{i,j} = (T_{\mathbf{n}}(h_{p_2} \otimes f_{p_1}))_{i,j} \left(\kappa_{1,1}(\hat{\tau}_i) - (\widetilde{D}_{\mathbf{n}}(\kappa_{1,1}))_{i,j} \right),$$

where from (2.11) and (5.4) we know

$$(\widetilde{D}_{\mathbf{n}}(\kappa_{1,1}))_{i,j} = \kappa_{1,1} \left(\frac{l_{\min(i,j)}}{n_1 + p_1 - 2}, \frac{k_{\min(i,j)}}{n_2 + p_2 - 2} \right),$$

$$\hat{\tau}_i = \xi_{l_i+2, k_i+2, [p_1, p_2]} = (\xi_{l_i+2, [p_1]}, \xi_{k_i+2, [p_2]}),$$

and

$$l_r := (r-1) \bmod (n_1 + p_1 - 2), \quad k_r := \left\lfloor \frac{r-1}{n_1 + p_1 - 2} \right\rfloor.$$

The structure of $T_{\mathbf{n}}(h_{p_2} \otimes f_{p_1}) = T_{n_2+p_2-2}(h_{p_2}) \otimes T_{n_1+p_1-2}(f_{p_1})$ and the band structure of $T_{n_2+p_2-2}(h_{p_2})$, $T_{n_1+p_1-2}(f_{p_1})$ imply that for the non-zero elements (5.8) at position (i, j) we have

$$|l_j - l_i| \leq \left\lfloor \frac{p_1}{2} \right\rfloor, \quad |k_j - k_i| \leq \left\lfloor \frac{p_2}{2} \right\rfloor.$$

Therefore, $\|T_{\mathbf{n}}(h_{p_2} \otimes f_{p_1})\|_\infty \leq C_T$, with C_T a constant independent of n . Moreover, by (4.24) we have

$$|(\widehat{L}_{n_1, n_2}^{[p_1, p_2]})_{i,j}| \leq |(T_{\mathbf{n}}(h_{p_2} \otimes f_{p_1}))_{i,j}| \omega \left(\kappa_{1,1}, \frac{C}{n} \right).$$

It follows that both $\|\widehat{L}_{n_1, n_2}^{[p_1, p_2]}\|_\infty$ and $\|(\widehat{L}_{n_1, n_2}^{[p_1, p_2]})^T\|_\infty$ are bounded above by

$$\omega \left(\kappa_{1,1}, \frac{C}{n} \right) \|T_{\mathbf{n}}(h_{p_2} \otimes f_{p_1})\|_\infty \leq C_T \omega \left(\kappa_{1,1}, \frac{C}{n} \right),$$

and so by (4.22) we have

$$\|\widehat{L}_{n_1, n_2}^{[p_1, p_2]}\| \leq \omega \left(\kappa_{1,1}, \frac{C}{n} \right) \|T_{\mathbf{n}}(h_{p_2} \otimes f_{p_1})\|_\infty \leq C_T \omega \left(\kappa_{1,1}, \frac{C}{n} \right).$$

This means that

$$\frac{\|\widehat{L}_{n_1, n_2}^{[p_1, p_2]}\|_1}{(\nu_1 n + p_1 - 2)(\nu_2 n + p_2 - 2)} \leq \|\widehat{L}_{n_1, n_2}^{[p_1, p_2]}\| \leq C_T \omega \left(\kappa_{1,1}, \frac{C}{n} \right).$$

Since $\kappa_{1,1}$ is continuous over the set $[0, 1]^2$, we deduce that $\lim_{n \rightarrow \infty} \omega \left(\kappa_{1,1}, \frac{C}{n} \right) = 0$, and this implies

$$\lim_{n \rightarrow \infty} \frac{\|\widehat{L}_{n_1, n_2}^{[p_1, p_2]}\|_1}{(\nu_1 n + p_1 - 2)(\nu_2 n + p_2 - 2)} = 0.$$

With similar arguments we can obtain analogous upper bounds for $\|\overline{L}_{n_1, n_2}^{[p_1, p_2]}\|$, $\|\overline{L}_{n_1, n_2}^{[p_1, p_2]}\|_1$, $\|\widetilde{L}_{n_1, n_2}^{[p_1, p_2]}\|$ and $\|\widetilde{L}_{n_1, n_2}^{[p_1, p_2]}\|_1$. Hence, all the hypotheses of Theorem 2.3 are satisfied, and we conclude that $\{\frac{1}{n^2} A_{n_1, n_2}^{[p_1, p_2]}\}_n$ is distributed in the sense of the eigenvalues as claimed. \square

The symbol in Theorem 5.1 can be compactly written in matrix form as

$$(5.9) \quad [\nu_1 \ \nu_2] (K \circ P_{p_1, p_2}) [\nu_1 \ \nu_2]^T,$$

where K is the coefficient matrix of our problem (5.1), and

$$(5.10) \quad P_{p_1, p_2} := \begin{bmatrix} h_{p_2} \otimes f_{p_1} & g_{p_2} \otimes g_{p_1} \\ g_{p_2} \otimes g_{p_1} & f_{p_2} \otimes h_{p_1} \end{bmatrix}.$$

Note that the spectral distribution described in Theorem 5.1 holds without any assumption on the coefficient matrix K except continuity. However, in order to ensure that (5.1) is an elliptic problem, this matrix has to be SPD.

Theorem 5.2. *The matrix P_{p_1, p_2} in (5.10) is Symmetric Positive Semi-Definite (SPSD) over the domain $[-\pi, \pi]^2$ and SPD for all (θ_1, θ_2) such that $\theta_1 \theta_2 \neq 0$. Moreover, if K is SPD then $K \circ P_{p_1, p_2}$ is SPD over $[-\pi, \pi]^2$ and SPD if $\theta_1 \theta_2 \neq 0$.*

Proof. From the construction of the matrix, it is clear that P_{p_1, p_2} is symmetric. Moreover, Lemmas 3.6 and 3.8 imply that

$$h_{p_2} \otimes f_{p_1} \geq 0, \quad f_{p_2} \otimes h_{p_1} \geq 0,$$

and if $\theta_1 \theta_2 \neq 0$ then

$$h_{p_2} \otimes f_{p_1} > 0, \quad f_{p_2} \otimes h_{p_1} > 0.$$

In addition, Lemma 3.9 ensures that

$$\det(P_{p_1, p_2}) = h_{p_2}(\theta_2) f_{p_2}(\theta_2) f_{p_1}(\theta_1) h_{p_1}(\theta_1) - (g_{p_2}(\theta_2))^2 (g_{p_1}(\theta_1))^2 \geq 0,$$

and if $\theta_1 \theta_2 \neq 0$ then $\det(P_{p_1, p_2}) > 0$. Thus, P_{p_1, p_2} is PSD over $[-\pi, \pi]^2$, and SPD if $\theta_1 \theta_2 \neq 0$.

Finally, we remark that the componentwise Hadamard product of two symmetric positive (semi-)definite matrices is a symmetric positive (semi-)definite matrix, because $A \circ B$ is a principal submatrix of $A \otimes B$. This concludes the proof. \square

From (3.16) and (3.34) we also obtain easily the following factorization of the matrix P_{p_1, p_2} .

Theorem 5.3. *Let P_{p_1, p_2} be defined as in (5.10), then*

$$P_{p_1, p_2} = S \hat{P}_{p_1, p_2} S,$$

with

$$S := \begin{bmatrix} 2 \sin(\theta_1/2) & 0 \\ 0 & 2 \sin(\theta_2/2) \end{bmatrix},$$

$$\hat{P}_{p_1, p_2} := \begin{bmatrix} h_{p_2}(\theta_2) h_{p_1-2}(\theta_1) & \hat{g}_{p_2}(\theta_2) \hat{g}_{p_1}(\theta_1) \\ \hat{g}_{p_2}(\theta_2) \hat{g}_{p_1}(\theta_1) & h_{p_2-2}(\theta_2) h_{p_1}(\theta_2) \end{bmatrix},$$

and $\hat{g}_p(\theta) := \frac{g_p(\theta)}{2 \sin(\theta/2)}$, for which holds

$$|2 \sin(\theta/2)|^p \left(\frac{1}{\theta^p} - \frac{1}{\pi^p} \right) \leq |\hat{g}_p(\theta)| \leq |2 \sin(\theta/2)|^p \frac{1}{\theta^p}.$$

The next theorem analyzes the zeros of the symbol in (5.9).

Theorem 5.4. *If K is SPD, the symbol in (5.9) is nonnegative over the domain $[-\pi, \pi]^2$ and has a unique zero of order two at $(\theta_1, \theta_2) = (0, 0)$.*

Proof. Theorem 5.2 implies that the matrix $K \circ P_{p_1, p_2}$ is SPSD over the domain $[-\pi, \pi]^2$, so the symbol in (5.9) is nonnegative on $[-\pi, \pi]^2$ and it can vanish only if $[\nu_1 \ \nu_2]^T$ is an eigenvector associated to a zero eigenvalue of $K \circ P_{p_1, p_2}$. From Theorem 5.2 we know that this can only occur if $\theta_1 \theta_2 = 0$.

From Theorem 5.3 and Lemmas 3.6–3.8 we may conclude that $\boldsymbol{\theta} := (\theta_1, \theta_2) = (0, 0)$ is a zero of order two for the symbol in (5.9). Indeed, by taking the Taylor expansion around $\boldsymbol{\theta} = (0, 0)$ of the matrix \hat{P}_{p_1, p_2} in Theorem 5.3 we have

$$\hat{P}_{p_1, p_2} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + O(\|\boldsymbol{\theta}\|),$$

so we obtain that the symbol is equal to

$$[\nu_1 \ \nu_2] S K S [\nu_1 \ \nu_2]^T + O(\|\boldsymbol{\theta}\|^3).$$

Since K is assumed to be SPD, the symbol has a zero of order two at $\boldsymbol{\theta} = (0, 0)$.

Moreover, it is easy to check that the matrix $K \circ P_{p_1, p_2}$ has a zero eigenvalue of multiplicity one if $\theta_1 = 0$ or $\theta_2 = 0$ but $(\theta_1, \theta_2) \neq (0, 0)$. In the first case, the second component of the corresponding eigenvector is zero. In the second case, the first component of the corresponding eigenvector is zero. Since $\nu_1 \nu_2 \neq 0$, it follows that in both cases $[\nu_1 \ \nu_2]^T$ cannot be an eigenvector associated to a zero eigenvalue of multiplicity one. Hence, $(\theta_1, \theta_2) = (0, 0)$ is the unique zero of (5.9) over the domain $[-\pi, \pi]^2$. \square

Remark 5.5. Theorem 5.4 states that the symbol in (5.9) has a unique (theoretical) zero at $(\theta_1, \theta_2) = (0, 0)$. However, other numerical zeros occur elsewhere for large $\mathbf{p} := (p_1, p_2)$. Indeed, from Lemmas 3.6–3.8 we can easily see that all entries in the matrix P_{p_1, p_2} vanish numerically at $\theta_1 = \pi$ or $\theta_2 = \pi$ for large values of \mathbf{p} . Since the entries in K are assumed to be continuous, it follows that the symbol in (5.9) has numerical zeros at $\theta_1 = \pi$ or $\theta_2 = \pi$ for large \mathbf{p} .

5.2. Spectral analysis with a geometry map. In order to address a general physical domain, and so a non-trivial geometry map \mathbf{G} , we can follow a similar approach as in Section 4.3 for the 1D case. The construction of the collocation matrices only requires some more tedious computations with respect to the 1D case due to the more involved expressions of the first and second derivatives of the inverse of \mathbf{G} . We omit the details and we refer to Appendix A for the computation of the derivatives of the basis functions φ_i defined in (2.7).

In view of the spectral analysis of the matrices we are interested in, it is sufficient to provide the expression of the second order terms for the transformed form of problem (5.1) in the presence of a geometry map,

$$\mathbf{G} : \hat{\Omega} \rightarrow \overline{\Omega}, \quad \mathbf{G}(\hat{x}, \hat{y}) = \begin{pmatrix} G_1(\hat{x}, \hat{y}) \\ G_2(\hat{x}, \hat{y}) \end{pmatrix}.$$

We denote by J the Jacobian of the geometry map \mathbf{G} and by $|J|$ its determinant, i.e.,

$$(5.11) \quad J := \begin{bmatrix} \frac{\partial G_1}{\partial \hat{x}} & \frac{\partial G_1}{\partial \hat{y}} \\ \frac{\partial G_2}{\partial \hat{x}} & \frac{\partial G_2}{\partial \hat{y}} \end{bmatrix}, \quad |J| := \frac{\partial G_1}{\partial \hat{x}} \frac{\partial G_2}{\partial \hat{y}} - \frac{\partial G_2}{\partial \hat{x}} \frac{\partial G_1}{\partial \hat{y}}.$$

Because $(x, y) = \mathbf{G}(\hat{x}, \hat{y})$, we have

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = \frac{1}{|J|} \begin{bmatrix} \frac{\partial G_2}{\partial \hat{y}} & -\frac{\partial G_2}{\partial \hat{x}} \\ -\frac{\partial G_1}{\partial \hat{y}} & \frac{\partial G_1}{\partial \hat{x}} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \hat{x}} \\ \frac{\partial}{\partial \hat{y}} \end{bmatrix} = J^{-T} \begin{bmatrix} \frac{\partial}{\partial \hat{x}} \\ \frac{\partial}{\partial \hat{y}} \end{bmatrix}.$$

Therefore,

$$\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} \kappa_{1,1} & \kappa_{1,2} \\ \kappa_{2,1} & \kappa_{2,2} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \hat{x}} \\ \frac{\partial}{\partial \hat{y}} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial \hat{x}} & \frac{\partial}{\partial \hat{y}} \end{bmatrix} J^{-1} \begin{bmatrix} \kappa_{1,1} & \kappa_{1,2} \\ \kappa_{2,1} & \kappa_{2,2} \end{bmatrix} J^{-T} \begin{bmatrix} \frac{\partial}{\partial \hat{x}} \\ \frac{\partial}{\partial \hat{y}} \end{bmatrix}.$$

Moreover, we have

$$\mathbf{1}(J^{-1}K(\mathbf{G})J^{-T} \circ Pu)\mathbf{1}^T = \nabla \cdot J^{-1}K(\mathbf{G})J^{-T}\nabla u \\ + \text{terms with lower order derivatives.}$$

It follows that the problem (2.1) is equivalent to

$$(5.12) \quad \begin{cases} -\mathbf{1}(J^{-1}K(\mathbf{G})J^{-T} \circ Pu)\mathbf{1}^T + \text{terms with lower order deriv.} = f(\mathbf{G}), & \text{in } \hat{\Omega}, \\ u = 0, & \text{on } \partial\hat{\Omega}. \end{cases}$$

The next theorem is a direct consequence of Theorem 5.1, and shows explicitly the influence of the geometry map on the spectral distribution of the collocation matrices for the problem (2.1), i.e., for the sequence $\{A_{n_1, n_2}^{[p_1, p_2]}\}_n$ obtained by applying the procedure described in Section 5.1 to the problem in (5.12).

Theorem 5.6. *Let $\mathbf{G} : \hat{\Omega} \rightarrow \bar{\Omega}$ be a geometry map such that $\mathbf{G} \in C^1(\hat{\Omega})$, \mathbf{G} is invertible in $\hat{\Omega}$ and has bounded second derivatives. Then, the sequence of matrices $\{\frac{1}{n^2}A_{n_1, n_2}^{[p_1, p_2]}\}_n$ (with $n_1 = \nu_1 n$ and $n_2 = \nu_2 n$) is distributed like the function*

$$(5.13) \quad [\nu_1 \ \nu_2] (J^{-1}K(\mathbf{G})J^{-T} \circ P_{p_1, p_2}) [\nu_1 \ \nu_2]^T,$$

in the sense of the eigenvalues, where the matrices P_{p_1, p_2} and J are defined in (5.10) and (5.11) respectively.

Remark 5.7. The expression of the symbol (5.9) and (5.13) has a completely similar structure as the expression of the differential problem (5.1) and (5.12) respectively. This motivates the reformulation of (1.1) into the less common form (2.1).

5.3. The d -dimensional setting. Following the same steps as in the 2D case, it is clear that no specific difficulties arise in the d -dimensional setting for any $d \geq 3$. In other words, all the relevant ingredients are already considered when passing from one dimension to two dimensions. We can summarize the general situation in the following way. Consider the matrices $\{\frac{1}{n^2}A_{n_1, \dots, n_d}^{[p_1, \dots, p_d]}\}_n$ (with $n_j = \nu_j n$, $j = 1, \dots, d$) approximating problem (2.1) in its full generality. Let $\mathbf{G} : \hat{\Omega} \rightarrow \bar{\Omega}$ such that $\mathbf{G} \in C^1(\hat{\Omega})$, \mathbf{G} is invertible in $\hat{\Omega}$ and with bounded second derivatives. Then, the sequence of matrices $\{\frac{1}{n^2}A_{n_1, \dots, n_d}^{[p_1, \dots, p_d]}\}_n$ (with $n_j = \nu_j n$, $j = 1, \dots, d$) is distributed like the function

$$(5.14) \quad \boldsymbol{\nu} (J^{-1}K(\mathbf{G})J^{-T} \circ P_{p_1, \dots, p_d}) \boldsymbol{\nu}^T,$$

in the sense of the eigenvalues. Here, $\boldsymbol{\nu} = [\nu_1 \cdots \nu_d]$ and the matrix $P_{p_1, \dots, p_d}(\boldsymbol{\theta})$ is the $d \times d$ symmetric matrix in the Fourier variables, whose entry in position (j, k) represents the formula approximating the second derivative $\frac{\partial^2}{\partial x_j \partial x_k}$, in analogy to the two-dimensional setting shown compactly in equation (5.10).

6. FINAL REMARKS AND CONCLUSIONS

We have presented a construction of the inherently non-symmetric matrices arising in isogeometric collocation methods based on uniform B-splines of degrees \mathbf{p} , approximating problem (2.1), and we have performed an analysis of their spectral properties. More specifically, we have computed the associated (spectral) symbol, that is the function describing their asymptotic spectral distribution (in the Weyl sense), when the matrix-size tends to infinity or, equivalently, the fineness parameters tend to zero. We have studied the symbol f : in accordance with previous results in FD and FE contexts (see [4, 24]), f has a canonical structure incorporating

- 1) the approximation technique, identified by a finite set of polynomials in the Fourier variables $\boldsymbol{\theta} := (\theta_1, \dots, \theta_d) \in [0, \pi]^d$;
- 2) the geometry, identified by the map \mathbf{G} in the variables $\hat{\mathbf{x}} := (\hat{x}_1, \dots, \hat{x}_d)$ defined on the reference domain $[0, 1]^d$;
- 3) the coefficients of the principal terms of the PDE, namely K in the physical variables $\mathbf{x} := (x_1, \dots, x_d)$ defined on the physical domain Ω .

Looking at the range of f , we have observed that f is a nonnegative function with a unique zero of order two at $\boldsymbol{\theta} = \mathbf{0}$, but it shows infinitely many numerical zeros for large $\|\mathbf{p}\|_\infty$ and in particular at $\theta_j = \pi$ if p_j is large. While the first fact is expected, because it is common in any approximation method (see analogous features in the FD, FE symbols [4, 24]), the second leads to the surprising fact that, for large $\|\mathbf{p}\|_\infty$, there is a subspace of high frequencies where the collocation matrices are ill-conditioned. This non-canonical feature is responsible for the slowdown, with respect to \mathbf{p} , of standard iterative methods. On the other hand, its knowledge and the knowledge of other properties of the symbol allow us to construct a preconditioned GMRES/multigrid method, for which optimal convergence properties are numerically observed, with a remarkable robustness with respect to all the relevant parameters: see the twin paper [12]. Still many issues remain, both on the theoretical and numerical sides, and these will be the subject of future research.

APPENDIX A. DERIVATIVES OF THE 2D IGA BASIS FUNCTIONS WITH A GEOMETRY MAP

Let $\mathbf{G} : \hat{\Omega} := [0, 1]^2 \rightarrow \bar{\Omega} \subset \mathbb{R}^2$ be a geometry map such that

$$(s, t) \in \hat{\Omega} \rightarrow \mathbf{G}(s, t) := (x(s, t), y(s, t)) \in \bar{\Omega}.$$

Then, we consider a basis function $\varphi_{i,j} : \bar{\Omega} \rightarrow \mathbb{R}$,

$$\varphi_{i,j}(x, y) := N_i(s(x, y))N_j(t(x, y)), \quad (s, t) = (s(x, y), t(x, y)) = \mathbf{G}^{-1}(x, y),$$

where we denote by N_l the univariate B-splines defined in Section 4. To simplify the notation, we omit the subscript corresponding to the degree and we use (s, t) as variables in $\hat{\Omega}$ instead of (\hat{x}, \hat{y}) . In addition we use the notation

$$\begin{aligned} x_s &:= \frac{\partial x}{\partial s}, \quad y_s := \frac{\partial y}{\partial s}, \quad x_t := \frac{\partial x}{\partial t}, \quad y_t := \frac{\partial y}{\partial t}, \\ s_x &:= \frac{\partial s}{\partial x}, \quad s_y := \frac{\partial s}{\partial y}, \quad t_x := \frac{\partial t}{\partial x}, \quad t_y := \frac{\partial t}{\partial y}. \end{aligned}$$

A similar notation will be used to denote derivatives of higher order. Finally, we denote by J the Jacobian of the map \mathbf{G} , see (5.11), and

$$|J| := x_s y_t - y_s x_t.$$

The first derivatives are given by

$$\frac{\partial \varphi_{i,j}}{\partial x} = N'_i N_j s_x + N_i N'_j t_x, \quad \frac{\partial \varphi_{i,j}}{\partial y} = N'_i N_j s_y + N_i N'_j t_y,$$

where

$$(A.1) \quad s_x = \frac{y_t}{|J|}, \quad t_x = -\frac{y_s}{|J|}, \quad s_y = -\frac{x_t}{|J|}, \quad t_y = \frac{x_s}{|J|}.$$

Note that

$$J^{-1} = \frac{1}{|J|} \begin{bmatrix} y_t & -x_t \\ -y_s & x_s \end{bmatrix} = \begin{bmatrix} s_x & s_y \\ t_x & t_y \end{bmatrix}.$$

The second derivatives are given by

$$\begin{aligned} \frac{\partial^2 \varphi_{i,j}}{\partial x^2} &= \frac{\partial}{\partial x} (N'_i N_j s_x + N_i N'_j t_x) \\ &= N''_i N_j s_x s_x + 2N'_i N'_j s_x t_x + N_i N''_j t_x t_x + N'_i N_j s_{xx} + N_i N'_j t_{xx}, \\ \frac{\partial^2 \varphi_{i,j}}{\partial y \partial x} &= \frac{\partial}{\partial y} (N'_i N_j s_x + N_i N'_j t_x) \\ &= N''_i N_j s_x s_y + N'_i N'_j s_x t_y + N'_i N'_j s_y t_x + N_i N''_j t_x t_y + N'_i N_j s_{xy} + N_i N'_j t_{xy}, \\ \frac{\partial^2 \varphi_{i,j}}{\partial y^2} &= \frac{\partial}{\partial y} (N'_i N_j s_y + N_i N'_j t_y) \\ &= N''_i N_j s_y s_y + 2N'_i N'_j s_y t_y + N_i N''_j t_y t_y + N'_i N_j s_{yy} + N_i N'_j t_{yy}. \end{aligned}$$

In the above expressions, we use

$$s_{xx} = \frac{\partial}{\partial x} s_x = \frac{\partial}{\partial x} \left(\frac{y_t}{|J|} \right) = \frac{\left(\frac{\partial}{\partial x} y_t \right) |J| - y_t \left(\frac{\partial}{\partial x} |J| \right)}{|J|^2},$$

and so

$$s_{xx} = \frac{(y_{ts} s_x + y_{tt} t_x) |J| - y_t \left(\frac{\partial}{\partial x} |J| \right)}{|J|^2}.$$

Similarly,

$$\begin{aligned} s_{yy} &= \frac{-(x_{ts} s_y + x_{tt} t_y) |J| + x_t \left(\frac{\partial}{\partial y} |J| \right)}{|J|^2}, \\ t_{xx} &= \frac{-(y_{ss} s_x + y_{st} t_x) |J| + y_s \left(\frac{\partial}{\partial x} |J| \right)}{|J|^2}, \\ t_{yy} &= \frac{(x_{ss} s_y + x_{st} t_y) |J| - x_s \left(\frac{\partial}{\partial y} |J| \right)}{|J|^2}, \end{aligned}$$

and finally,

$$\begin{aligned}\frac{\partial}{\partial x}|J| &= (x_{ss}y_t + x_sy_{ts})s_x + (x_{st}y_t + x_sy_{tt})t_x \\ &\quad - (y_{ss}x_t + y_sx_{ts})s_x - (y_{st}x_t + y_sx_{tt})t_x, \\ \frac{\partial}{\partial y}|J| &= (x_{ss}y_t + x_sy_{ts})s_y + (x_{st}y_t + x_sy_{tt})t_y \\ &\quad - (y_{ss}x_t + y_sx_{ts})s_y - (y_{st}x_t + y_sx_{tt})t_y.\end{aligned}$$

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